2026/2/25 RFCMP

moonshine現象の紹介

(Introduction to moonshine phenomena)

RFCMP 2026 Mini Workshop

moonshine 現象の紹介　岡田 昌樹

https://masakiokada0101.github.io/talks.html 補足資料

1. Preliminaries from group theory

1.1. Finite simple groups

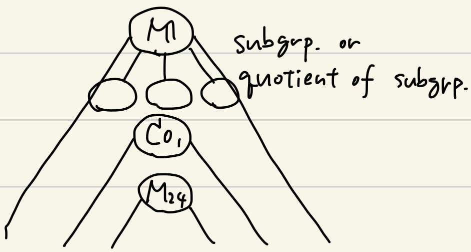
Jordan-Hölder theorem

no normal subgrps. other than G and $\{1\}$

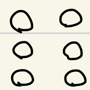
Thm (Classification of fin. simple grps.)

A fin. simple grp. is one of the followings.

- cyclic grp. of prime order $\mathbb{Z}_p$ (p : prime)
- alternating grp. A_n $(n \geq 5)$
- simple grps. of Lie type
- 26 sporadic grps.



6 pariahs (not subgrp. / quotient of subgrp. of $\mathbb{M}$)



Three important fin. simple grps. :

				order
the Mathieu grp. M_{24}	$=$ $\mathrm{Aut}(G_{24})$	$=$	?	2.45×10^{8}
the Conway grp. Co_1	$=$ $\mathrm{Aut}(\Lambda_{24}) / \mathbb{Z}_2$	$=$	$\mathrm{Aut}_{N=1}(V^{f\natural})$	4.16×10^{18}
the moster grp. $\mathbb{M}$	$=$ $\mathrm{Aut}(B^{\natural})$	$=$	$\mathrm{Aut}(V^{\natural})$	8.08×10^{53}

$B^{\natural} = V^{\natural}|_{wt=2}$ Griess alg.

· Conway grps.

Def

A <u>lattice</u> $(L, \langle -,- \rangle)$ of $\begin{matrix}\text{dim.}\\ \text{rk.}\end{matrix}$ n ($L \cong$)
is a free ab. grp. $L \subset \mathbb{R}^n$ of rk. n whose basis spans $\mathbb{R}^n$
with a non-dgn. sym. bilin form $\langle -,- \rangle : L \times L \to \mathbb{R}$.

A 24-dim. pos-def. even self-dual lattice is called a <u>Niemeier lattice</u>.

<u>The Leech lattice Λ_{24}</u> is the Niemeier lattice without vectors with $|k|^2 = 2$.
(unique up to isom.)

$Co_0 := \mathrm{Aut}(\Lambda_{24})$ $\qquad$ $\mathrm{Center}(Co_0) = \{\pm 1\} \cong \mathbb{Z}_2$

$Co_1 := Co_0/\mathbb{Z}_2$ $\qquad$ simple

· moster grp. $\mathbb{M}$

Fact $\mathbb{M} = \langle 2^{1+24}.Co_1, \sigma \rangle$, $\quad \sigma^2 = 1$.

group extension

Def

G is an extraspecial p-group p^{1+n} (p : prime)

$\overset{\text{def}}{\iff} \begin{cases} \cdot\ \mathrm{Center}(G) \cong \mathbb{Z}_p \\ \cdot\ G/\mathrm{Center}(G) \cong (\mathbb{Z}_p)^n \end{cases}$

ex. gamma matrix algebra

$$\langle \gamma_1, \cdots, \gamma_{2m} \mid \{\gamma_i, \gamma_j\} = -2\delta_{ij} \rangle$$

If the number of γ's is odd $\langle \gamma_1, \cdots, \gamma_{2m+1} \rangle$, then $\pm \gamma_1 \cdots \gamma_{2m+1} \in$ center.

$$\cup$$

$$\{\pm 1, \pm\gamma_i, \pm\gamma_i\gamma_j, \cdots, \pm\gamma_1\cdots\gamma_{2m}\} \cong 2^{1+2m}$$

center $= \{\pm 1\}$

1.2. Group extension

how to combine two groups N and G.
generalization of $N \times G$, $N \rtimes G$.

Def

G, N : grps.

An extension of G by N is a short exact seq. of grp. homs.

$$1 \to N \hookrightarrow \hat{G} \underset{\text{section } s}{\overset{\pi}{\rightleftarrows}} G \to 1 .$$

($\hat{G}$ is also called an extension. Any extension is denoted by $N.G$.)

- $N \triangleleft \hat{G}$
- $\hat{G}/N \cong G$

Take a section $\underbrace{s : G \to \hat{G}}_{\pi \circ s = id_G \text{ not necessarily a grp. hom.}}$. $N \times G \to \hat{G}$ is a bijection.

$$(n, g) \mapsto n \cdot s(g)$$

- If we can take a grp-hom. section s (If the short ex. seq. splits),

then the multiplication law of $\hat{G}$ is $\pi(s(g) n' s(g)^{-1}) = g \cdot 1 \cdot g^{-1} = 1$ $\therefore s(g) n' s(g)^{-1} \in N$

$$(n \cdot s(g)) \cdot (n' \cdot s(g')) = (n \cdot \underbrace{s(g) n' s(g)^{-1}}) \cdot s(gg')$$

$$=: (n \cdot \varphi_g(n')) \cdot s(gg')$$

where $\varphi_g : N \to N$ defines a G-action $G \curvearrowright N$.

$\therefore$ $\hat{G} \cong N \rtimes_\varphi G$. $N:G$

In particular, if φ is a trivial action $\varphi_g = id_N$, then $\hat{G} \cong N \times G$.

· Suppose s is not a grp. hom.

We can define $\varepsilon : G \times G \to N$ s.t.

$$s(g)\, s(g') = \varepsilon(g, g')\, s(gg').$$

$\pi(\underbrace{s(g)s(g')s(gg')^{-1}}_{\in N}) = 1$

Let N be abelian. Then $\begin{array}{rccc} \varphi_g : & N & \to & N \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & n & \mapsto & s(g)\, n\, s(g)^{-1} \end{array}$ defines a G-action $G \curvearrowright N$.

$s(g')s(g)\, n\, s(g)^{-1}s(g')^{-1}$
$= s(g'g)\, n\, s(g'g)^{-1}$
$\uparrow$
$s(g'g)^{-1}s(g')s(g) \in N$: abelian

The multi. law of $\hat{G}$ is

$$(n \cdot s(g)) \cdot (n' \cdot s(g')) = (n \cdot \varphi_g(n') \cdot \varepsilon(g, g')) \cdot s(gg').$$

ε is a 2-cocycle :

$$\varepsilon(g, g') \cdot \varepsilon(gg', g'') = \varphi_g(\varepsilon(g', g''))\, \varepsilon(g, g'g'').$$

$\leftarrow$ associativity of $\hat{G}$

Prop

G : a grp.

N : a G-module (an ab. grp. with G-action)

$$\left\{\begin{array}{l}\text{extensions of } G \text{ by } N \\ \text{compatible with the } G\text{-action}\end{array}\right\} / \text{equiv.} \quad \overset{1:1}{\longleftrightarrow} \quad H^2(G; N)$$

$$\left(\begin{array}{l}\text{A set } N \times G \text{ with multi. law} \\ (n, g) \cdot (n', g') = (n \cdot g(n') \cdot \varepsilon(g, g'), gg')\end{array}\right) \longleftarrow [\varepsilon]$$

$$N \rtimes G \longleftarrow 0$$

※ Even if $\hat{G} \cong \hat{G}'$, they can be inequiv. as grp. extensions.

$$\begin{tikzcd} & \hat{G} \arrow[dd, "\wr"] \arrow[dr] & & \\ 1 \arrow[r] & N \arrow[u] \arrow[d] & G \arrow[r] & 1 \\ & \hat{G}' \arrow[ur] & & \end{tikzcd}$$

In particular, if $N \subset \mathrm{Center}(\hat{G})$, then φ is a trivial action.

$$(n \cdot s(g)) \cdot (n' \cdot s(g')) = (n \cdot n' \cdot \varepsilon(g, g')) \cdot s(gg').$$

$\hat{G}$ is called a <u>central extension</u>.

Examples

- Fact $H^2(Co_1 ; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (trivial action)

  $\rightsquigarrow$ There are only two central extensions : $\underset{\text{split}}{\mathbb{Z}_2 \times Co_1}$ and $\underset{\text{non-split}}{Co_0}$

- carry (繰り上がり) central extension, non-split

$$\underset{0-99}{\mathbb{Z}_{100}} \cong \underset{十の位}{\mathbb{Z}_{10}} . \underset{一の位}{\mathbb{Z}_{10}}$$

$27+38=65 \qquad (2,7)+(3,8) = (2+3+\underbrace{\varepsilon(7,8)}_{=1\ (\text{carry})}, 7+8) = (6,5)$

2. Monstrous moonshine

modular j-function

· modular inv.
(a mod. func. of weight 0)

$$j_1(\tau) = q^{-1} + \underbrace{196884}_{\substack{\shortparallel \\ 1+196883}} q + \underbrace{21493760}_{\substack{\shortparallel \\ 1+196883+21296876}} q^2 + \cdots \quad (q = e^{2\pi i\tau})$$

$$j(\tau) = q^{-1} + 744 + \cdots$$

McKay (1978), Thompson (1979)

dim. of irrep. of $\mathbb{M}$

$$d_1 = 1, \quad d_2 = 196883, \quad d_3 = 21296876, \quad \cdots$$

"monstrous moonshine" Conway Norton

Moreover, if we decompose $j_1(\tau)$ into the irr. characters

$$\mathrm{ch}_h(\tau) := \mathrm{Tr}_{V_h}\, q^{L_0 - \frac{c}{24}} \quad (V_h : \text{the irr. highest-weight rep. of weight } h)$$

of the Virasoro alg. of $c = 24$,

$$j_1(\tau) = d_1\,\mathrm{ch}_0 + d_2\,\mathrm{ch}_2 + d_3\,\mathrm{ch}_3 + d_4\,\mathrm{ch}_4$$
$$+ d_6\,\mathrm{ch}_5 + (d_5 + d_7)\,\mathrm{ch}_6 + \cdots$$

very simple!

This observation was explained by a CFT / VOA with $\mathbb{M}$ symmetry.

idea A chiral CFT / VOA V is a rep. sp. of the Vir. alg.:

$$V = V_{h_1} \oplus \cdots \oplus V_{h_1} \oplus V_{h_2} \oplus \cdots \oplus V_{h_2} \oplus \cdots$$

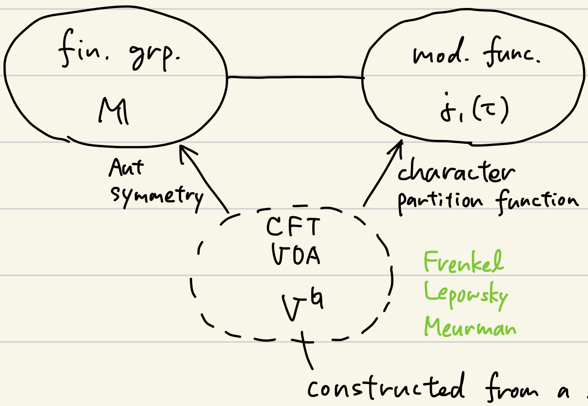
If V has the symmetry $\mathbb{M}$ ($\exists$ action $\mathbb{M} \curvearrowright V$ which commutes with the Vir. action) ($\mathrm{Aut}(V) \supset \mathbb{M}$),

$$V = W_1 \otimes V_{h_1} \oplus W_2 \otimes V_{h_2} \oplus \cdots$$

rep. sp. of $\mathbb{M}$

Its character is

$$\mathrm{Tr}_V\, q^{L_0 - \frac{c}{24}} = \sum_i (\dim W_i)\, \mathrm{ch}_{h_i}(\tau)$$



Many other relations

fin. grp. — mod. func. / weak Jacobi form

are observed.

· lattice VOA

pos-def. even self-dual lattice L of rk. n

$\rightsquigarrow$ bosonic mod-inv. VOA V_L of $c = n$

creation operator $[\alpha^i_m, \alpha^j_n] = m\,\delta_{ij}\,\delta_{m+n,0}$

As a vec. sp., $V_L = \mathrm{Span}_{\mathbb{C}} \left\{ \underbrace{\alpha^{i_1}_{-m_1} \cdots \alpha^{i_\ell}_{-m_\ell} |k\rangle}_{\substack{\text{weight} \\ (L_0\text{-eigenvalue})}} \,\middle|\, \begin{array}{l} 1 \le i_j \le n, \; \ell = 0, 1, \cdots, \\ m_j \in \mathbb{Z}_{>0}, \; k \in L \end{array} \right\}$

weight (L_0-eigenvalue) $m_1 + \cdots + m_\ell + \frac{1}{2}|k|^2$

Its character is

$$\mathrm{Tr}_{V_L} q^{L_0 - \frac{c}{24}} = \frac{\Theta_L(\tau)}{\eta(\tau)^n}$$

$$\overset{n=24}{=} j_1(\tau) + \underbrace{\cancel{24}}_{\alpha^i_{-1}|0\rangle} + \underbrace{\cancel{\#\{k \in L \mid |k|^2 = 2\}}}_{|k\rangle}$$

$L_0 - \frac{c}{24} = 0$

Orbifold by $\mathbb{Z}_2$
$(X^i \mapsto -X^i)$

$L = \Lambda_{24}$

So, we consider $V^\natural := V_{\Lambda_{24}} / \mathbb{Z}_2$.

monster VOA (moonshine module)

· How $\mathrm{Aut}(V^\natural) = \mathbb{M}$?

$$\Lambda_{24} \qquad\qquad \mathrm{Aut}(\Lambda_{24}) = Co_0$$

cocycle factor $\varepsilon : L \times L \to \mathbb{Z}_2$ $\rightsquigarrow$

$$\mathrm{Hom}(L,\mathbb{Z}_2) \cong (\mathbb{Z}_2)^{24}$$

central extension $1 \to \mathbb{Z}_2 \to \hat{\Lambda}_{24} \to \Lambda_{24} \to 0 \qquad \mathrm{Aut}(\hat{\Lambda}_{24}) = 2^{24}.Co_0$

$\uparrow$ non-split

add cr.-an. op. $\rightsquigarrow$

$$V_{\Lambda_{24}} \qquad\qquad \mathrm{Aut}(V_{\Lambda_{24}}) = 2^{24}.Co_0$$

$\mathbb{Z}_2$-orbifold: add $\mathbb{Z}_2$-twisted sector $\rightsquigarrow$

$$V_{\Lambda_{24}} \oplus (V_{\Lambda_{24}})_{tw} \qquad\qquad \curvearrowleft\ 2^{1+24}.Co_0$$

project onto $\mathbb{Z}_2$-inv. states $\downarrow$

$$V^\natural \qquad\qquad \curvearrowleft\ 2^{1+24}.Co_1$$

$\uparrow$ This action does not mix the untw. sector and the tw. sector.

$$(V_L)^{\mathbb{Z}_2\text{-inv}} \oplus (V_L)_{tw}^{\mathbb{Z}_2\text{-inv.}}$$

$$\circlearrowright\ V_0 = (V_L)^{\mathbb{Z}_2\text{-inv}}, \qquad T_0 = (V_L)_{tw}^{\mathbb{Z}_2\text{-inv.}}\ \circlearrowleft$$

$$V_0 \oplus V_3, \qquad T_0 \oplus T_3, \qquad V_3 \overset{\sigma}{\longleftrightarrow} T_3$$

σ triality operator

Thm (Frenkel, Lepowsky, Meurman) "Vertex operator algebras and the Monster" (1988)

$$\mathrm{Aut}(V^\natural) = \langle 2^{1+24}.Co_1\,,\ \sigma \rangle \cong \mathbb{M}.$$

3. Conway (super)moonshine

3.1. Duncan's module

We will construct an $\mathcal{N}=1$ VOSA $V^{f\natural}$ of $c=12$ and $V^{f\natural}|_{\text{weight }\frac{1}{2}}=0$.

"Conway supermoonshine module"
"Duncan's module"

Thm (Duncan, 2007)

$\mathrm{Aut}_{\mathcal{N}=1}(V^{f\natural}) \cong Co_1$

Thm (Duncan, Mack-Crane, 2014)

The uniqueness of $V^\natural$ is still a conj.
$(c=24,\ V^\natural|_1=0)$ (FLM uniqueness conj.)

Such a "VOSA" of $c=12$ without weight-$\frac{1}{2}$ states is unique up to isom.

a self-dual C_2-cofinite rational VOSA of CFT type

· n real chiral free fermions

NS sector (Clifford module VOSA)

As a vec. sp.,

$$F_{NS} = \mathrm{Span}_{\mathbb{C}}\Big\{ \underbrace{\psi^{i_1}_{-r_1}\cdots\psi^{i_\ell}_{-r_\ell}|0\rangle}_{\text{weight } r_1+\cdots+r_\ell} \ \Big|\ \begin{matrix} 1\le i_j\le n, & \ell=0,1,2,\cdots \\ r_j\in(\mathbb{Z}+\frac{1}{2})_{>0}, & \{\psi^i_r,\psi^j_s\}=-2\delta^{ij}\delta_{r+s,0}\end{matrix} \Big\}.$$

$\{\psi^i_{-r},\psi^j_{-s}\}=0$ ($\hat{0}$, $\hat{0}$)

R sector (canonically-twisted module) n: even

$\Psi^i_0 := \frac{1}{\sqrt{2}}(\psi^{2i-1}_0+\sqrt{-1}\,\psi^{2i}_0)$, $\bar{\Psi}^i_0 := \frac{1}{\sqrt{2}}(\psi^{2i-1}_0-\sqrt{-1}\,\psi^{2i}_0)$ ← choice of the polarization

As a vec. sp.,

$$F_R = \mathrm{Span}_{\mathbb{C}}\Big\{ \underbrace{\psi^{i_1}_{-r_1}\cdots\psi^{i_\ell}_{-r_\ell}\,\Psi^{i'_1}_0\cdots\Psi^{i'_{\ell'}}_0|0\rangle_R}_{\text{weight } r_1+\cdots+r_\ell+\frac{n}{16}} \ \Big|\ \begin{matrix} 1\le i_j\le n,\ 1\le i'_j\le\frac{n}{2}, & \ell,\ell'=0,1,2,\cdots \\ r_j\in\mathbb{Z}_{>0}, & \{\psi^i_r,\psi^j_s\}=-2\delta^{ij}\delta_{r+s,0}\end{matrix}\Big\}.$$

· Spin(n) action

$\psi^1, \cdots, \psi^n$: an orthonormal $\mathbb{C}$-basis of $V \cong \mathbb{C}^n$

$(\psi^i+\psi^j)\otimes(\psi^i+\psi^j) = -|\psi^i|^2 - |\psi^j|^2 - 2\langle\psi^i,\psi^j\rangle$

$\psi^i\otimes\psi^j + \psi^j\otimes\psi^i = -2\delta^{ij}$

$$\mathrm{Cliff}(V) := \bigoplus_{i=0}^{\infty} V^{\otimes i} \Big/ v\otimes v \sim -|v|^2 \qquad \{\psi^i, \psi^j\} = -2\delta^{ij}$$

$\mathrm{Spin}(n) \subset \mathrm{Cliff}(V)$

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}_2 & \to & \mathrm{Spin}(n) & \twoheadrightarrow & SO(n) & \to & 1 \\ & & & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ & & & & \pm\hat{M} & \longmapsto & M & & \end{array}$$

a double cover of $SO(n)$

$$\hat{M}(\psi^1 \cdots \psi^n)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\hat{M}^{-1} = (\psi^1 \cdots \psi^n)\, M \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$x \in \mathrm{Spin}(n)$ acts on

F_{NS} as $\psi^{i_1}_{-r_1}\cdots\psi^{i_\ell}_{-r_\ell}|0\rangle \mapsto (x\psi^{i_1}x^{-1})_{-r_1}\cdots(x\psi^{i_\ell}x^{-1})_{-r_\ell}|0\rangle$ $SO(n)$-action

F_R as $\psi^{i_1}_{-r_1}\cdots\psi^{i_\ell}_{-r_\ell}\Psi^{i_1'}_0\cdots\Psi^{i_{\ell'}}_0|0\rangle_R \mapsto (x\psi^{i_1}x^{-1})_{-r_1}\cdots(x\Psi^{i_{\ell'}}x^{-1})_0\, \underset{\substack{\uparrow \\ \mathrm{Cliff}(V) \\ \cong \\ \langle\psi^1_0,\cdots,\psi^n_0\rangle}}{x}|0\rangle_R$

$\langle\psi^1_{-r},\cdots,\psi^n_{-r}\rangle \not\cong$

When n is even, $-1_{SO(n)} \in SO(n)$.

The lifts of $-1_{SO(n)}$ are $\pm\psi^1\cdots\psi^n =: \pm z$.

$$\mathrm{Center}(\mathrm{Spin}(n)) = \begin{cases} \mathbb{Z}_2 = \langle -1_{\mathrm{Spin}(n)} \rangle & (n:\text{odd}) \\ \mathbb{Z}_4 = \langle z \rangle & (n \equiv 2 \mod 4) \\ \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle -1_{\mathrm{Spin}(n)} \rangle \times \langle z \rangle & (n \equiv 0 \mod 4) \end{cases}$$

If $n \equiv 0 \mod 4$,

$$\begin{array}{ccc} & Spin(n) & \\ /\langle \mathfrak{z} \rangle \swarrow & & \searrow /\langle -1_{Spin(n)} \rangle \\ SemiSpin(n) & \overset{n \neq 8}{\not\cong} & SO(n) \end{array}$$

$$\left(\begin{array}{ccc} & \begin{array}{c|c} \hat{M} & -\hat{M} \\ \hline \mathfrak{z}\hat{M} & -\mathfrak{z}\hat{M} \end{array} & \\ \swarrow & & \searrow \\ m \mid -m & & \begin{array}{c} M \\ \hline -M \end{array} \end{array} \right)$$

parities of each sector :

fermion parity $\mathfrak{z}$ \ $-1_{Spin(n)}$	$+1$	-1	
$+1$	F^0_{NS}	F^0_R	$\overset{n=24}{=\!=\!=}: V^{f\natural}$ (NS) $\curvearrowleft$ SemiSpin(24)
-1	F^1_{NS}	F^1_R	$=\!=\!=: V^{f\natural}_{tw}$ (R)
	$\Vert$ F_{NS} $\circlearrowleft$ $SO(n)$	$\Vert$ F_R	

($-\mathfrak{z} \rightsquigarrow$ different parity of R sector) just a convention

"$\langle \mathfrak{z} \rangle$-orbifold"

"$\langle -1_{Spin(n)} \rangle$-orbifold"

minimum-weight states of each sector :

$F^0_{NS} \ni |0\rangle$ 1-dim. wt. 0 $\qquad F^0_R \ni$ (even # of Ψ^i_0's) $|0\rangle_R$ $2^{\prime\prime}$-dim. wt. $\frac{3}{2}$ positive chiral rep. of Spin(24)

$F^1_{NS} \ni \psi^i_{-\frac{1}{2}}|0\rangle$ 24-dim. wt. $\frac{1}{2}$ $\qquad F^1_R \ni$ (odd —〃—) $|0\rangle_R$ $2^{\prime\prime}$-dim. wt. $\frac{3}{2}$ negative —〃—

No weight-$\frac{1}{2}$ states in $V^{f\natural}$.

· Co_1 action on $V^{f\natural}$

Fact

$$
\begin{tikzcd}
 & \exists\, Co_0 \subset \mathrm{Spin}(24) \arrow[dl, "/\langle z \rangle"'] \arrow[dr, phantom, "\cong"] & \\
\mathrm{SemiSpin}(24) \supset Co_1 & & Co_0 = \mathrm{Aut}(\Lambda_{24}) \subset SO(24)
\end{tikzcd}
$$

$\circlearrowright$ $V^{f\natural}$ Fact $\det = +1$

In the 2^{11}-dim. weight-$\frac{3}{2}$ subsp. $(V^{f\natural})_{\frac{3}{2}}$,

there exists a 1-dim. inv. subsp. under the Co_1 action.

Its basis G constitutes the supercurrent of the Co_1-inv. $\mathcal{N}=1$ SCA.

$\longrightarrow \mathrm{Aut}_{\mathcal{N}=1}(V^{f\natural}) \cong Co_1$.

Stolz-Teichner? proposal? program?

3.2. Connection to the Segal-Stolz-Teichner conjecture

Conj cf. [1108.0189 Stolz-Teichner]

$$\mathrm{SQFT}_\bullet \cong \mathrm{TMF}_\bullet$$

the space ($\mathbb{Z}$-graded ring) of the classes of topological modular forms

the space ($\mathbb{Z}$-graded ring) of the equiv. classes of 2d $\mathcal{N}=(0,1)$ SQFTs

For $[\mathcal{T}] \in \mathrm{SQFT}_n$, degree

- the anomaly polynomial of the grav. anom. $= \frac{n}{48} p_1$ (first Pontrjagin class)
- $\mathrm{SQFT}_n \xrightarrow{\sim} \mathrm{TMF}_n \longrightarrow (\mathrm{MF}_{\mathbb{Z}}^{w\text{-}h})_{\frac{n}{2}}$ weakly holomorphic modular forms with $\mathbb{Z}$ coefficient of weight $\frac{n}{2}$

$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{\frac{n}{2}} f(\tau)$

$$[\mathcal{T}] \longmapsto \eta(\tau)^n Z_{ell}[\mathcal{T}](\tau)$$

$$Z_{ell}[\mathcal{T}](\tau) := \mathrm{Tr}_{\mathcal{H}_{\mathcal{T},\text{R-sector}}} (-1)^F q^{\frac{H+P}{2}} \bar{q}^{\frac{H-P}{2}}$$

If $\mathcal{T}$ is an SCFT,

- $n = 2(c_R - c_L)$
- $Z_{ell}[\mathcal{T}](\tau) = \mathrm{Tr}_{\mathcal{H}_{\mathcal{T},\text{R-sector}}} (-1)^{F_L+F_R} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}}$

indep. of $\bar{q}$ (the function of q)

If we assume that the SST conj. is true, we can obtain nontrivial statements on $\mathrm{SQFT}_\bullet$ by translating the properties of $\mathrm{TMF}_\bullet$.

· Example 1. the divisibility property

$\mathrm{TMF}_n \longrightarrow (\mathrm{MF}_{\mathbb{Z}}^{w\text{-}h})_{\frac{n}{2}}$ is not surjective.

Its image is also determined. (Hopkins, Mahowald) Hopkins cf. [math/0212397 Prop.4.6]

In particular,

$$(\text{its image}) \cap \mathbb{Z}[\eta(\tau)] = \begin{cases} \mathbb{Z}\left[\frac{24}{\gcd(24,\frac{n}{24})}\eta(\tau)^n\right] & (n \equiv 0 \mod 24) \\ 0 & (\text{otherwise}) \end{cases}$$

Therefore, if the SST conj. is true, then for $[\mathcal{T}] \in \mathrm{SQFT}_n$,

$$Z_{\mathrm{ell}}[\mathcal{T}](\tau) \text{ is const.} \Rightarrow \begin{cases} \frac{24}{\gcd(24,\frac{n}{24})} \mid Z_{\mathrm{ell}}[\mathcal{T}](\tau) & (24 \mid n) \\ Z_{\mathrm{ell}}[\mathcal{T}](\tau) = 0 & (\text{otherwise}) \end{cases}$$

For example,

$$V^{f\natural} : (c_L, c_R) = (12, 0) \quad n = -24$$

$$\frac{24}{\gcd(24,\frac{n}{24})} = 24 \mid Z_{\mathrm{ell}}[V^{f\natural}](\tau) = -24$$

· Example 2. the periodicity element

Fact

There exists a periodicity element $X \in TMF_{-24^2=-576}$ s.t.

$$X\cdot : TMF_n \xrightarrow{\sim} TMF_{n-576}.$$

Under the SST conj., this translates to the existence of

an $\mathcal{N}=(0,1)$ SQFT $\mathcal{T}$ of $n=-576$ s.t. $Z_{ell}[\mathcal{T}]=1$.

It suffices to find

an $\mathcal{N}=1$ chiral SCFT $\mathcal{T}$ of $c=288$ s.t. $Z_{ell}[\mathcal{T}]=1$.

$\downarrow$

$Z_{ell}[\mathcal{T}](\tau)=\text{const.}$

$(V^{f\natural})^{\otimes 24}$ has $c = 24\times 12 = 288$.

[1907.08388 Johnson-Freyd]

The anomaly of its Co_1 sym. is described by the generator of $SH(Co_1)\cong \mathbb{Z}_{24}$.

$\rightsquigarrow$ $(V^{f\natural})^{\otimes 24}/Co_1$ has $Z_{ell}[\mathcal{T}]=1$?

$(V^{f\natural})^{\otimes 24}/(Co_1\times A_{24})$ $\xrightarrow{\quad /\!/ \quad}$? [2210.14923 Albert, Lin, Kaidi]

$Co_1\times S_{24}$ is anomalous.

4. Mathieu moonshine

K3 surface : a cpx-2-dim. $\overbrace{\text{CY}}^{\text{cpt. Kähler mfd. with triv. can. bdl.}}$ mfd. with $h^{0,1} = 0$.
$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$

K3-target SCFT : $\mathcal{N} = (4,4)$, $(c_L, c_R) = (6,6)$

The elliptic genus   the weak Jacobi form of weight 0 and index $\frac{c_L}{6} = 1$

$$Z_{K3}(\tau, z) := \mathrm{Tr}_{\mathcal{H}_{K3}} (-1)^{F_L + F_R} y^{J_0} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} \qquad \left(\begin{array}{l} q = e^{2\pi i \tau} \\ y = e^{2\pi i z} \end{array} \right)$$

$$= 2\varphi_{0,1}(\tau, z)$$

$$= 2\left\{ y + 10 + y^{-1} + (10y^2 - 64y + 108 - 64y^{-1} + 10y^{-2})q + \cdots \right\}$$

If we decompose Z_{K3} into the irr. char. of the $\mathcal{N}=4$ SCA of $c=6$,

$$Z_{K3}(\tau, z) = 20\, \mathrm{ch}^{(\text{massless})}_{\frac{1}{4}, 0}(\tau, z) - 2\,\mathrm{ch}^{(\text{massless})}_{\frac{1}{4}, \frac{1}{2}}(\tau, z)$$
$$+ 2 \sum_{n=1}^{\infty} A_n\, \mathrm{ch}_{n+\frac{1}{4}, \frac{1}{2}}(\tau, z) .$$

n	1	2	3	4	5	6	$\cdots$
A_n	45	231	770	2277	5796	13915	$\cdots$

$13915 = 3520 + 10395$

dim. of irrep. of M_{24} :

1, 23, 45, 45, 231, 231, 252, 253, 483, 770, 770,
990, 990, 1035, 1035, 1035, 1265, 1771, 2024,
2277, 3312, 3520, 5313, 5796, 5544, 10395

$\longrightarrow$ underlying VOA-like object ?

K3 Mathieu moonshine (江口, 大栗, 立川, 2010)

other evidence of the existence of VOA

- twined ell. genera (the McKay-Thompson series) Cheng (2010) Gaberdiel, Hohenegger, Volpato (2010)

  $\exists$ M_{24}-module reproducing them. Gannon (2012)

- twisted twined ell. genera

  Their mod. trsf. are controlled by a 3-cocycle in $H^3(M_{24}; U(1))$.

  (fit into the theory of anomaly) Gaberdiel, Persson, Ronellenfitsch, Volpato (2012)

However, preserve the hol. 2-form

- $\mathrm{Aut}_\omega(K3) \subsetneq M_{23}$ Mukai (1988)
- $M_{24} \not\subset \mathrm{Aut}_{\mathcal{N}=(4,4)}(K3\ \mathrm{CFT}) \subsetneq Co_0$ Gaberdiel, Hohenegger, Volpato (2010)

The underlying VOA-like object is not yet found!

solution?

- symmetry surfing? Taormina, Wendland (2011-13)
- reflected K3 $= V^{f\natural}$? Duncan, Mack-Crane (2015) Taormina, Wendland (2017)
- preserve only $\mathcal{N}=(4,1)$? Harvey, Moore (2020)

  some subalgebra?

- non-inv. sym.? Volpato (2024) Angius, Giaccari, Harrison, Volpato (2025)

  ⋮