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Supplementary Notes on "Introduction to Moonshine Phenomena"

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Introduction

1 Introduction

Unexpected coincidences between different fields which have been studied independently often occur in the history of science, and they promote the development and understanding of both fields. One notable example is moonshine phenomena.

In finite group theory in mathematics, the classification of finite simple groups was a long-standing problem. Nowadays, it is already settled, and finite simple groups are classified into three infinite series, and 26 sporadic groups. In 1978, when the existence of the largest sporadic group called the monster group $\mathbb{M}$ was still a conjecture, McKay noticed that its smallest nontrivial irreducible representation dimension 196883 appears in the coefficient of the modular j-function

$$j(\tau) = q^{-1} + 774 + 196884q + \cdots \quad (q = e^{2\pi\sqrt{-1}\tau}), \tag{1.1}$$

as $196884 = 196883 + 1$ [McK01]. Furthermore, Thompson observed that the first few coefficients of the modular j-function can also be written as simple sums of irreducible representation dimensions of the monster group $\mathbb{M}$ [Tho79b]. It came as a surprise to mathematicians at that time, because the modular j-function is a concept important in arithmetic geometry of elliptic curves, a field studied independently of finite group theory. Conway and Norton deepened the relation between the monster group $\mathbb{M}$ and the modular j-function, and proposed a conjecture which they call the monstrous moonshine [CN79]. Here, the English word "moonshine" means "insubstantial or unreal" [Gan06a].

The appearance of the irreducible representations of the monster group in the modular j-function was theoretically explained by Frenkel, Lepowsky, and Meurman. They constructed an algebraic object called a vertex operator algebra (VOA), whose automorphism group is the monster group, and whose graded character is the modular j-function (up to the constant term) [FLM88]. However, this moonshine phenomenon was not confined to mathematics. Surprisingly, the concept of vertex operator algebras provided a mathematical formulation of two-dimensional conformal field theories in physics. Conformal field theory (CFT) is an important framework in physics, which describes the string theory in high-energy physics and the critical phenomena in condensed matter physics.

This triggered off a lot of interaction between mathematics and physics. For example, many other moonshine phenomena were observed between the sporadic groups and modular functions or weak Jacobi forms, and one of the prominent examples, the K3 Mathieu moonshine, was found by physicists Eguchi, Ooguri, and Tachikawa [EH10]. It has been studied by both mathematicians and physicists to this day, but its mysterious nature has not yet been fully understood.

Moreover, conformal field theories with sporadic group symmetries have been extending beyond moonshine phenomena. For example, they are studied in [Wit07, GGK+08] in relation to three-dimensional gravity. In addition, a specific VOA called the Conway moonshine module is expected to be useful in providing supporting evidence for the Stolz–Teichner conjecture, which also proposes a new correspondence between mathematics and physics [ST11].

Part I

Review of Basic Concepts

This Part I is a review of basic concepts related to moonshine phenomena. A moonshine phenomenon is, simply put, an observed relationship between a finite group, in particular a simple one, and a modular function. So we review some general facts about finite simple groups in Section 2, and modular functions in Section 3. The most classical example of moonshine phenomena, the monstrous moonshine, was theoretically explained by an underlying object called a vertex operator algebra (VOA). VOAs are also expected to be key ingredients in understanding other moonshine phenomena, and provide connections with two-dimensional conformal field theories (CFT) in physics. We will review the axiomatic definition of a VOA in Section 4 to get the idea of how VOAs mathematically formulate CFTs in physics, but the definition itself is somewhat technical. In many cases for physicists, it suffices to consider the examples of VOAs reviewed in the next Part II, whose constructions would be more familiar to physicists working with CFTs.

2 Finite Simple Groups

In this Section 2, we review basic facts about finite simple groups, mainly focusing on important examples of sporadic finite simple groups. After mentioning the classification of finite simple groups in Section 2.1, we will describe the Mathieu groups, the Conway groups, and the monster group in Sections 2.2, 2.3, and 2.4, respectively. Along the way, we also review fundamental concepts such as codes and lattices, partly in order to fix definitions and notations; we mainly follow the definitions in [CS99].

Before proceeding, let us clarify the convention of permutation groups and its actions, and some notations regarding groups and lattice vectors here.

Convention of permutation groups and its actions

The symmetric group S_n consists of permutations $\sigma : \Omega_n \to \Omega_n$ where $\Omega_n = \{1, \ldots, n\}$, and there are two conventions for the definition of the multiplication:

$$(1,2) \cdot (1,3) = (1,3,2);\ 1 \mapsto 3, 3 \mapsto 2, 2 \mapsto 1, \tag{2.1}$$

$$(1,2) \tilde{\cdot} (1,3) = (1,2,3);\ 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1. \tag{2.2}$$

They are related as $\tau \cdot \sigma = \sigma \tilde{\cdot} \tau$.

In these notes, we will adopt the multiplication $\cdot$ (2.1) and the left action of $\sigma \in S_n$ on a vector $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$ as

$$\sigma(k) = (k_{\sigma^{-1}(1)}, \ldots, k_{\sigma^{-1}(n)}). \tag{2.3}$$

This can also be written as $(\sigma(k))_{\sigma(i)} = k_i$.

In GAP [GAP22] and the webpage of ATLAS of Finite Group Representations [WWT$^+$], on the other hand, they adopt the multiplication $\tilde{\cdot}$ (2.2). So when we cite equations from them, we will convert them into representations in terms of the multiplication $\cdot$ (2.1) in these notes.

Notations of groups

For groups G and N, following ATLAS of Finite Groups [CCN$^+$85], we write $N.G$ for any extension of G by N.

$$1 \to N \to N.G \to G \to 1. \tag{2.4}$$

If it is a split extension, or equivalently a semidirect product, then we write $N : G$ or $N \rtimes G$. Furthermore, if it is a direct product, then we write $N \times G$. We sometimes use the notation $N^{\cdot}G$ for a non-split extension. For readers not familiar with these topics, Appendix A contains an elementary introduction to group extensions.

For a group G, we let $H^n(G; A)$ denote the n-th group cohomology[1] of G with coefficients in A. Here, A is a G-module, although in most cases in these notes, the G-action on A is trivial.

$R^\times$ denotes the multiplicative group of a ring R.

$\mathbb{K}[G]$ denotes the group algebra of G over $\mathbb{K}$.

The cyclic group $\mathbb{Z}_n$ is sometimes denoted by just n.

The *general linear group* $\mathrm{GL}(n, \mathbb{K})$ is the group of all the invertible $n \times n$ matrices over the field $\mathbb{K}$. The *special linear group* $\mathrm{SL}(n, \mathbb{K})$ is the subgroup of $\mathrm{GL}(n, \mathbb{K})$ of all the matrices with $\det = 1$. The *projective general linear group* $\mathrm{PGL}(n, \mathbb{K})$ is the quotient group of $\mathrm{GL}(n, \mathbb{K})$ by its center $\mathbb{K}^\times \cdot 1$, which consists of the non-zero scalar matrices. The *projective special linear group* $\mathrm{PSL}(n, \mathbb{K})$ is the quotient group of $\mathrm{SL}(n, \mathbb{K})$ by its center, which consists of the non-zero scalar matrices with $\det = 1$. In some literature, $\mathrm{PGL}(n, \mathbb{Z}_q)$ is also denoted by $\mathrm{PGL}(n, q)$, and $\mathrm{PSL}(n, \mathbb{Z}_q)$ is by $\mathrm{PSL}(n, q)$ or $L_n(q)$.

1_G or simply 1 denotes the identity element of G. I denotes the identity matrix. id_X denotes the identity map $X \to X$.

Notations of lattice vectors

For a lattice vector $k \in L \subset \mathbb{R}^n$ of a lattice L, the components of k with respect to the standard basis of $\mathbb{R}^n$ are denoted by subscript as $k = (k_1, \ldots, k_n)$, and the components of k with respect to another basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ (for example a $\mathbb{Z}$-basis of L, but not limited to such a case) are denoted by superscript as $k = \sum_i k^i e_i$.

We let $\mathbb{e}_1, \ldots, \mathbb{e}_n$ denote an orthonormal basis of $\mathbb{R}^n$ where the lattice L is embedded, with respect to the symmetric bilinear form of L (extended by $\mathbb{R}$-linearity).

When the basis is obvious from the context, $\vec{1}$ denotes $(1, \ldots, 1)$.

[1] There is another concept, the cohomology $H^n_{\mathrm{top}}(X; A)$ of a topological space X. In this notation, the group cohomology $H^n(G; A)$ in the main text is the cohomology $H^n_{\mathrm{top}}(BG; A)$ of the classifying space BG of G.

2.1 Classification of Finite Simple Groups

A group G is said to be *simple* if its normal subgroups are the trivial ones $\{1\}$ and G only. A *composition series* of a group G is a finite series $\{1\} = H_0 \subset H_1 \subset \cdots \subset H_n = G$, where H_{i-1} is a proper normal subgroup of H_i, and H_i/H_{i-1} is simple, for all $i = 1, \ldots, n$. The groups H_i/H_{i-1} are called the *composition factors*, and n is called the *length* of the composition series. If a group has a composition series, it is unique in the following sense.

Theorem 2.1 (the Jordan–Hölder theorem)**.** *If a group has a composition series, then any two composition series of it have the same length, and the composition factors of them are the same up to permutation and isomorphism.*

It is known that any finite group has a composition series. In this sense, finite simple groups are fundamental building blocks of finite groups.

The classification of finite simple groups was achieved through the tremendous efforts of many mathematicians.

Theorem 2.2 (classification of finite simple groups)**.** *Every finite simple group is isomorphic to one of the following groups:*

- *cyclic groups $\mathbb{Z}_p$ of prime order,*
- *alternating groups A_n of degree $n \geq 5$,*
- *finite simple groups of Lie type,*
- *26 sporadic groups, listed in Table 2.1.*

Among the 26 sporadic groups, the largest one is the monster group $\mathbb{M}$, and 20 sporadic groups (including $\mathbb{M}$) are involved in $\mathbb{M}$ as subgroups or quotients of subgroups. They are called the *happy family*, and the other 6 sporadic groups are called *pariahs* in [Gri82]. The happy family is further classified into three generations [Gri82]: the first generation is involved in the Mathieu group M_{24}, the second generation the Conway group Co_1, and the third generation the monster group $\mathbb{M}$.

Table 2.1: The list of sporadic groups. For each sporadic group G, we also show its name, order $|G|$, number $c(G)$ of conjugacy classes, Schur multiplier $H_2(G;\mathbb{Z})$, which is isomorphic to $H^2(G;\mathrm{U}(1))$ if G is finite [Gri98, (2.12.2)], and generation as the happy family, where "P" denotes a pariah. The notation of sporadic groups and data are taken from [CCN+85].[2]

G	name	$\lvert G\rvert$		$c(G)$	$H^2(G;\mathrm{U}(1))$	generation
M_{11}	Mathieu	$2^4 3^2 5.11$	$= 7920$	10	1	1
M_{12}	Mathieu	$2^6 3^3 5.11$	$= 95040$	15	$\mathbb{Z}_2$	1
J_1	Janko	$2^3 3.5.7.11.19$	$= 175560$	15	1	P
M_{22}	Mathieu	$2^7 3^2 5.7.11$	$= 443520$	12	$\mathbb{Z}_{12}$	1
HJ, J_2	Hall–Janko	$2^7 3^3 5^2 7$	$= 604800$	21	$\mathbb{Z}_2$	2
M_{23}	Mathieu	$2^7 3^2 5.7.11.23$	$\sim 1.02 \times 10^7$	17	1	1
HS	Higman–Sims	$2^9 3^2 5^3 7.11$	$\sim 4.44 \times 10^7$	24	$\mathbb{Z}_2$	2
J_3	Janko	$2^7 3^5 5.17.19$	$\sim 5.02 \times 10^7$	21	$\mathbb{Z}_3$	P
M_{24}	Mathieu	$2^{10} 3^3 5.7.11.23$	$\sim 2.45 \times 10^8$	26	1	1
McL	McLaughlin	$2^7 3^6 5^3 7.11$	$\sim 8.98 \times 10^8$	24	$\mathbb{Z}_3$	2
He	Held	$2^{10} 3^3 5^2 7^3 17$	$\sim 4.03 \times 10^9$	33	1	3
Ru	Rudvalis	$2^{14} 3^3 5^3 7.13.29$	$\sim 1.46 \times 10^{11}$	36	$\mathbb{Z}_2$	P
Suz	Suzuki	$2^{13} 3^7 5^2 7.11.13$	$\sim 4.48 \times 10^{11}$	43	$\mathbb{Z}_6$	2
$O'N$	O'Nan	$2^9 3^4 5.7^3 11.19.31$	$\sim 4.61 \times 10^{11}$	30	$\mathbb{Z}_3$	P
Co_3	Conway	$2^{10} 3^7 5^3 7.11.23$	$\sim 4.96 \times 10^{11}$	42	1	2
Co_2	Conway	$2^{18} 3^6 5^3 7.11.23$	$\sim 4.23 \times 10^{13}$	60	1	2
Fi_{22}	Fischer	$2^{17} 3^9 5^2 7.11.13$	$\sim 6.46 \times 10^{13}$	65	$\mathbb{Z}_6$	3
HN	Harada–Norton	$2^{14} 3^6 5^6 7.11.19$	$\sim 2.73 \times 10^{14}$	54	1	3
Ly	Lyons	$2^8 3^7 5^6 7.11.31.37.67$	$\sim 5.18 \times 10^{16}$	53	1	P
Th	Thompson	$2^{15} 3^{10} 5^3 7^2 13.19.31$	$\sim 9.07 \times 10^{16}$	48	1	3
Fi_{23}	Fischer	$2^{18} 3^{13} 5^2 7.11.13.17.23$	$\sim 4.09 \times 10^{18}$	98	1	3
Co_1	Conway	$2^{21} 3^9 5^4 7^2 11.13.23$	$\sim 4.16 \times 10^{18}$	101	$\mathbb{Z}_2$	2
J_4	Janko	$2^{21} 3^3 5.7.11^3 23.29.31.37.43$	$\sim 8.68 \times 10^{19}$	62	1	P
Fi'_{24}	Fischer	$2^{21} 3^{16} 5^2 7^3 11.13.17.23.29$	$\sim 1.26 \times 10^{24}$	108	$\mathbb{Z}_3$	3
B	baby monster (Fisher)	$2^{41} 3^{13} 5^6 7^2 11.13.17.19.23.31.47$	$\sim 4.15 \times 10^{33}$	184	$\mathbb{Z}_2$	3
$\mathbb{M}$	monster (Fisher–Griess)	$2^{46} 3^{20} 5^9 7^6 11^2 13^3 17.19.23.29.31.41.47.59.71$	$\sim 8.08 \times 10^{53}$	194	1	3

[2] There are also lists of sporadic groups in [Gri98,Har99,Yos24]. Note that some orders $|G|$ on the lists of [Gri98,Yos24] are different from those on [CCN+85], and seem to be typos.

Historically, the sporadic groups first constructed were the Mathieu groups M_{11}, M_{12} in 1861 [Mat61], and M_{22}, M_{23}, M_{24} in 1873 [Mat73]. The problem of classifying all the finite simple groups was questioned in an article [Hö92] by Hölder in 1892. Chevalley and others constructed simple groups called the groups of Lie type around 1955 [Che55], but the advent of a new sporadic group had to await the discovery of the Janko group J_1, announced in a one-page paper [Jan65] in 1965, and further clarified in [Jan66]. This was the kick-off of the successive discoveries of all the sporadic groups, which coincidentally ended again with the Janko group J_4 in 1976 [Jan76]. Then the finite group theory was oriented toward the completion of the classification theorem. It was declared that the proof was completed in 1981 [Gor82], but it was established as an accumulation of numerous results scattered throughout the literature, so Gorenstein, Lyons, and Solomon launched a project (the GLS project) to simplify the proof and to present it as a streamlined series of books. However, a serious gap was found in the classification of groups called quasithin groups. It was in 2004 that the gap was finally closed by Aschbacher and Smith in their two-volume books [AS04b, AS04a], as many as 1221 pages in total. Nowadays, the classification of finite simple groups is regarded as an established theorem. The books of the GLS project have reached volume 10 [CGLS23] in 2023, and the project is still ongoing.

As for the history of classification of finite simple groups, see [Asc94, §15], [Sol01, Gur18, Sol18a], [Asc04], and in particular for the GLS project, see [Sol18b]. References on sporadic groups include [Asc94], [Gri98], and [CS99]. ATLAS of Finite Groups [CCN$^+$85] is a standard database of finite simple groups, and there is also the webpage ATLAS of Finite Group Representations [WWT$^+$]. Regarding the literature on sporadic groups written in Japanese, the unpublished lecture notes[3] [Kon96] by Kondo and several expository articles were the only available references for a long time, although there was a book [Har99] by Harada focusing on the monster group and the monstrous moonshine. Fortunately, the situation has drastically improved with the publication of Yoshiara's book [Yos24] in 2024.

In the following sections, we will review particularly important sporadic groups, the Mathieu groups, the Conway groups, and the monster group. They can be described as the automorphism groups (or their subgroups or quotients) of some algebraic objects, the binary Golay code, the Leech lattice, and the Griess algebra or the monster VOA, respectively.

2.2 The Mathieu Groups and the Golay Codes

There are several ways to define or construct the Mathieu groups, and one of the most approachable ways is to define the largest Mathieu group M_{24} as the automorphism group of the binary Golay code G_{24}. So we first review generalities of linear codes and the binary Golay code in Section 2.2.1, and then introduce the Mathieu groups in Section 2.2.2. As for more details of the Mathieu groups and their different constructions, see for example [Gri98, CS99, Iva18, Yos24, Cur25].

[3] The author thanks Masahiko Miyamoto for sharing these notes.

2.2.1 The Golay Codes

Basic definitions for codes

A *q-ary linear code* $\mathcal{C}$ of *dimension* m and *length* n is an m-dimensional subspace of the n-dimensional $\mathbb{F}_q$-linear space $(\mathbb{F}_q)^n$, where q is a prime or a prime power and $\mathbb{F}_q$ is the finite field of order q. In these notes, codes always refer to linear ones. An element of a code is called a *codeword*, and the *(Hamming) weight* of a codeword $w = (w_1, \ldots, w_n)$ is $\mathrm{wt}(w) := |\{i \mid w_i \neq 0\}|$. When the minimal nonzero weight $\min_{w\in\mathcal{C}\setminus\{0\}} \mathrm{wt}(w)$ is d, this linear code is denoted by $[n, m, d]_q$.

Two linear codes are said to be *equivalent* when one is mapped to the other by a *monomial* matrix, which is a matrix containing exactly one nonzero element of $\mathbb{F}_q$ in each row and column. An equivalent map from a code to itself is called an *automorphism*, and the set of all the automorphisms of $\mathcal{C}$ forms the *automorphism group* $\mathrm{Aut}(\mathcal{C})$ of $\mathcal{C}$. Be careful that some literature calls the quotient of $\mathrm{Aut}(\mathcal{C})$ by its center $\mathbb{F}_q^\times \cdot 1$, which consists of the non-zero scalar matrices, the automorphism group of $\mathcal{C}$ (see e.g. [AM66]).

When $q = p^a$ with p prime, the *dual code* $\mathcal{C}^*$ of $\mathcal{C}$ is defined by

$$\mathcal{C}^* := \{v \in (\mathbb{F}_q)^n \mid v \cdot \bar{w} = 0 \text{ for any } w \in \mathcal{C}\}, \tag{2.5}$$

where $\bar{w} := ((w_1)^p, \ldots, (w_n)^p)$ is the *conjugate* of w, and $v \cdot w := \sum_i v_i w_i$. A code $\mathcal{C}$ is said to be *self-dual* if $\mathcal{C}^* = \mathcal{C}$. Since $\dim \mathcal{C}^* = n - m$, the length of a self-dual code must be even.

A binary code is said to be *even* if the weight of any codeword is even, and *doubly-even* if a multiple of 4. An even but not doubly-even code is said to be *singly-even*. A singly-even self-dual code is sometimes called *Type I*, and a doubly-even self-dual code *Type II*. A Type II code exists if and only if the length is a multiple of 8 [CS99, Ch. 7 §6 Cor. 18].

The binary and ternary Golay codes

The *(extended) binary Golay code* G_{24} is the unique $[24, 12, 8]_2$ linear code up to equivalence. This code is doubly-even and self-dual. A basis of G_{24} can be read off from the proof of [CS99, Ch. 10 §2.1 Thm. 7] as

$$\begin{aligned}
&(1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0),\\
&(0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0),\\
&(0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,0,0),\\
&(0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,0),\\
&(0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0),\\
&(0,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0),\\
&(0,0,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0,0),\\
&(0,0,0,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0,0),\\
&(0,0,0,0,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0,0),\\
&(0,0,0,0,0,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0,0),\\
&(0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,1,0),\\
&(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1).
\end{aligned} \tag{2.6}$$

The 24 columns are labeled by $0, 1, \ldots, 22, \infty$ in [CS99], which we will write $1, 2, \ldots, 24$ below.

The binary Golay code has 1, 759, 2576, 759, and 1 codewords with weight 0, 8, 12, 16, and 24, respectively. Codewords with weight 8 and 12 are called an *octad* and a *dodecad*, respectively.

The code obtained by dropping one coordinate of G_{24} is called the *(perfect) binary Golay code* G_{23}. We omit "extended" or "perfect" when it is obvious from the context. The perfect Golay code G_{23} is the unique $[23, 12, 7]_2$ linear code up to equivalence. The extended Golay code G_{24} can be recovered from G_{23} by adding a parity bit (a bit which makes the weight of the codeword even).

The *perfect ternary Golay code* G_{11} and the *extended ternary Golay code* G_{12} are respectively the unique $[11, 6, 5]_3$ and $[12, 6, 6]_3$ linear codes up to equivalence. G_{11} can be obtained by dropping one coordinate of G_{12}, and G_{12} can be recovered from G_{11} by adding a zero-sum check digit [CS99, (2.8.5)]. G_{12} is self-dual.

The proof of uniqueness of these four Golay codes G_{24}, G_{23}, G_{12}, and G_{11} are comprehensively explained in [MS77, Ch. 20], based on the original papers [Ple68, DG75]. [vL99, §4.2] and [Ple98, §10] also contain some proofs.

2.2.2 The Mathieu Groups

The largest Mathieu group M_{24}

The automorphism group $\mathrm{Aut}(G_{24})$ of a binary Golay code is the largest *Mathieu group* M_{24}. It is a sporadic simple group, and 5-transitive as a subgroup of S_{24} acting on 24 points. Here,

Definition 2.3 (k-transitive)**.** An action of a group G on a set X is said to be *k-transitive* (*k-fold transitive*) if for any k elements $\{i_1, \ldots, i_k\} \subset X$ and k elements $\{j_1, \ldots, j_k\} \subset X$, there is $g \in G$ such that $g \cdot i_t = j_t$. In addition, if such g is unique for any $\{i_1, \ldots, i_k\} \subset X$ and $\{j_1, \ldots, j_k\} \subset X$, then the action is said to be *sharply k-transitive*. A 1-transitive action is just called a *transitive* action.

A group G is said to be *k-transitive*, *sharply k-transitive*, if G is isomorphic to a subgroup of S_n such that its action on $\Omega_n = \{1, \ldots, n\}$ is k-transitive, sharply k-transitive, respectively. ■

Obviously, a k-transitive group is also k'-transitive for $k' \leq k$.

In the above basis (2.6), M_{24} is generated by the following four permutations [CS99, Ch. 10 §2.1]:

$$\begin{aligned}
&(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23),\\
&(16, 8, 15, 6, 11, 21, 18, 12, 23, 22, 20)(4, 7, 13, 2, 3, 5, 9, 17, 10, 19, 14),\\
&(24, 1)(16, 4)(8, 14)(15, 19)(6, 10)(11, 17)(21, 9)(18, 5)(12, 3)(23, 2)(22, 13)(20, 7),\\
&(15, 18, 12, 20, 23)(21, 11, 8, 6, 22)(19, 5, 3, 7, 2)(9, 17, 14, 10, 13).
\end{aligned} \tag{2.7}$$

A presentation of M_{24} can be found in [WWT$^+$] as

$$M_{24} = \langle a, b \mid a^2 = b^3 = (ba)^{23} = [b^{-1}, a]^{12} = [(bab)^{-1}, a]^5 = (b^{-1}ab^{-1}aba)^3(b^{-1}ababa)^3 = ((b^{-1}aba)^3ba)^4 = 1\rangle, \tag{2.8}$$

where $[y^{-1}, x^{-1}] = y^{-1}x^{-1}yx$ and note that the multiplication (2.2) in [WWT+] is converted to (2.1) here. In M_{24} generated by (2.7), these generators a and b can be taken as[4]

$$a = (1,8)(2,7)(3,19)(4,23)(5,20)(6,15)(9,16)(10,11)(12,22)(13,18)(14,24)(17,21), \tag{2.9}$$

$$b = (2,9,12)(3,13,24)(4,17,21)(5,16,15)(8,10,23)(11,14,22). \tag{2.10}$$

The second largest Mathieu group M_{23}

The *Mathieu Group* M_{23} is the stabilizer group of one point of the action of M_{24} on 24 points. Since M_{24} is transitive, M_{23} is unique up to isomorphism, regardless of the choice of the stabilized point. Equivalently, M_{23} is the automorphism group $\mathrm{Aut}(G_{23})$ of the perfect binary Golay code G_{23}. It is also a sporadic simple group.

A presentation of M_{23} can be found in [WWT+] as

$$M_{23} = \langle a, b \mid a^2 = b^4 = (ba)^{23} = (b^2a)^6 = [b^{-1}, a]^6 = (b^2ab^{-1}aba)^4 = 1, \\ (b^{-1}a)^3(ba)^3(b^{-1}aba)^2b^2ab^{-1}a(ba)^3 = b^2ab^{-1}abab^2aba(b^{-1}ab^2a)^2(b^2aba)^3 = 1\rangle. \tag{2.11}$$

If we take M_{23} as the subgroup of M_{24} generated by (2.7) stabilizing the point 23, these generators a and b can be taken as

$$a = (1,15)(2,20)(3,12)(4,19)(5,17)(7,13)(8,18)(11,14), \tag{2.12}$$

$$b = (1,12,17,21)(2,6,3,8)(4,9,16,20)(5,23)(10,15,22,13)(11,19). \tag{2.13}$$

More on Mathieu groups

There are five sporadic Mathieu groups: M_{24}, M_{23}, M_{22}, M_{12}, and M_{11}.

We already introduced M_{24} and M_{23}. M_{22} is the stabilizer group of two points (not as a set; each point must be stabilized) of the action of M_{24} on 24 points. Since M_{24} is 2-transitive, M_{22} is unique up to isomorphism, regardless of the choice of the stabilized points. In a similar manner, since M_{24} is at most 5-transitive, we can also define M_{21}, M_{20}, and M_{19}. M_{21} is simple, but not sporadic because it is isomorphic to $\mathrm{PSL}(3, \mathbb{F}_4)$. $M_{20} \cong 2^4 : A_5$ and $M_{19} \cong 2^4 : 3$ are not simple [Yos24, §4.1].

M_{12} is the quotient $\mathrm{Aut}(G_{12})/\langle -1\rangle$ of the automorphism group of the extended ternary Golay code by its center $\langle -1\rangle = \mathbb{Z}_2$. $\mathrm{Aut}(G_{12}) \cong \mathbb{Z}^2.M_{12}$ is a non-split extension. M_{12} is sharply 5-transitive as a subgroup of S_{12}. M_{11} is the point stabilizer of M_{12}. The automorphism group $\mathrm{Aut}(G_{11})$ of the perfect ternary Golay code is $\mathbb{Z}_2 \times M_{11}$ [AM66, Theorem 1].

Thanks to the classification of finite simple groups, multiply transitive finite groups are also classified [DM96, §7]. Obviously, S_n is sharply n-transitive, and A_n is sharply $(n-2)$-transitive.

[4]These generators (2.9) and (2.10) were found in the result of the GAP [GAP22] command `IsomorphismGroups(MathieuGroup(24), G)`, where `G` is declared as a `Group` generated by the permutations (2.7). This command returns one explicit isomorphism in the form of a map between generators, so it suffices to check the relations in (2.8) for the displayed generators.

Except for them, there are no k-transitive finite groups for $k \geq 6$. M_{12} is the only sharply 5-transitive finite group, M_{24} is the only non-sharply 5-transitive one, M_{11} is the only sharply 4-transitive (and not 5-transitive) one, and M_{23} is the only non-sharply 4-transitive (and not 5-transitive) one, except for S_n and A_n. M_{22} is 3-transitive, but there are other 3-transitive finite groups, for example $\mathrm{PSL}(2, \mathbb{F}_q)$.

We have introduced the Mathieu groups as the automorphism groups of the Golay codes (or their subgroups or quotients), but there is another well-established construction as the automorphism groups of the Steiner system.

Definition 2.4 (the Steiner system)**.** The *Steiner system* $S(t, k, n)$ ($1 \leq t \leq k \leq n$) is a set $\{B_1, \ldots, B_b\}$ of k-element subsets $B_i \subset \Omega_n$ of $\Omega_n = \{1, \ldots, n\}$, such that for any t-element subset $T \subset \Omega_n$, there is a unique $1 \leq i \leq b$ such that $T \subset B_i$. ■

The Steiner system does not necessarily exist for a given (t, k, n), and if exists, $b = \binom{n}{t}/\binom{k}{t}$, because the number $\binom{n}{t}$ of t-element subsets must be equal to $b \cdot \binom{k}{t}$. It is known that the automorphism groups of $S(5, 8, 24)$, $S(4, 7, 23)$, and $S(3, 6, 22)$ are M_{24}, M_{23}, and $M_{22}.2$, respectively. It is also known that the automorphism groups of $S(5, 6, 12)$ and $S(4, 5, 11)$ are M_{12} and M_{11}, respectively.

The Schur multipliers $H^2(G; \mathrm{U}(1))$ of the Mathieu groups M_{24}, M_{23}, M_{11} were correctly calculated to be trivial in [BF66]. Those of M_{22}, M_{12} were also calculated there, but turned out to be wrong, and corrected in [BF68, Maz82] to be the cyclic groups of order 12, 2, respectively.

2.3 The Conway Groups and the Leech Lattice

The Conway groups are defined as the automorphism group (or its subgroups or quotients) of the Leech lattice Λ_{24}. So we first review generalities of lattices and the Leech lattice in Section 2.3.1, and then we introduce the Conway groups in Section 2.3.2. As for the facts cited here and for more on the Leech lattice and the Conway groups, see for example [Gri98, Ch. 9], [Yos24, Sec. 7]. [CS99] contains comprehensive explanations and data on lattices.

2.3.1 The Leech Lattice and the Odd Leech Lattice

Basic definitions for lattices

A *lattice* of rank n is a free abelian group L of rank n whose basis is an $\mathbb{R}$-basis of a vector space $\mathbb{R}^n$ with a symmetric bilinear form $\langle -, - \rangle : L \times L \to \mathbb{R}$. Such a lattice is denoted by the pair $(L, \langle -, - \rangle)$, or just L, and naturally regarded as a subset of $\mathbb{R}^n$. We will only consider the cases where the symmetric bilinear forms are non-degenerate. If the symmetric bilinear form is positive-definite, then the lattice is said to be *positive-definite* or *Euclidean*. If the symmetric bilinear form is indefinite, and of signature (r, s), then the lattice is said to be *indefinite* or *Lorentzian*, and of *signature* (r, s).

A vector k in L is sometimes called a *lattice point* of L, and we also write the product $\langle k, k' \rangle$ of two vectors $k, k' \in L$ given by the symmetric bilinear form as $k \cdot k'$. If any vectors $k, k' \in L$ satisfy $k \cdot k' \in \mathbb{Z}$, then the lattice is said to be *integral*. The *squared length* of a vector $k \in L$ is defined as $|k|^2 := k \cdot k$. A vector $k \in L$ with $|k|^2 \in 2\mathbb{Z}$ is called an *even vector*, and $k \in L$ with $|k|^2 \in 2\mathbb{Z} + 1$ is called an *odd vector*. An integral lattice is said to be *even* if its vectors are all even, and *odd* otherwise. If we take a $\mathbb{Z}$-basis $e_1, \dots, e_n$ of L, then the matrix $G = [e_i \cdot e_j]_{i,j}$ is called the *Gram matrix* of L.

An *isometry* or *isomorphism* $g : L \to L'$ of lattices L and L' is an isomorphism of free abelian groups compatible with their symmetric bilinear forms. The group of all the isometries $L \to L$, or the *automorphisms* of L, is denoted by $\mathrm{Aut}(L)$ or $O(L)$, but note that $\mathrm{Aut}(L)$ sometimes denotes the automorphism group of just a free abelian group L (e.g. Section 5.2.3).

The *dual lattice* L^* of L is defined by

$$L^* := \{l \in \mathbb{R}^n \mid \langle l, k \rangle \in \mathbb{Z} \text{ for any } k \in L\}, \tag{2.14}$$

with the same symmetric bilinear form as that of L. L is integral if and only if $L \subset L^*$. L is said to be *self-dual* or *unimodular* if $L^* = L$. An odd self-dual lattice is called *Type I*, and an even self-dual lattice is called *Type II*.

Similarly to Type II codes, a Type II positive-definite lattice exists if and only if the rank is a multiple of 8 [CS99, Ch. 7 §6 Cor. 18]. Up to isometry, the E_8 lattice is the only one of rank 8. $E_8 \oplus E_8$ and D_{16}^+ are the only ones of rank 16. There are 24 Type II positive-definite lattices of rank 24, and they are called the *Niemeier lattices* [Nie73]. 23 lattices of them have vectors with squared length 2, whereas the last one, the Leech lattice Λ_{24}, does not. Moreover, a Type II indefinite lattice of signature (r, s) $(r, s > 0)$ exists if and only if $r - s \equiv 0 \bmod 8$, and it is unique up to isometry [Ser73, Ch. V].

Root lattices

A positive-definite even lattice generated by vectors with squared length 2 is called a *root lattice*.[5] Any root lattice is known to be a direct sum of the *irreducible* root lattices, which are

[5] Note that there is also the concept of a *root* of a semisimple Lie algebra $\mathfrak{g}$, whose squared length is not necessarily 2, and the *root lattice* $L_\mathfrak{g}$ of $\mathfrak{g}$, which is the lattice generated by the roots of $\mathfrak{g}$. If all the roots have squared length 2, then $\mathfrak{g}$ is said to be *simply-laced*.

The subgroup $W_\mathfrak{g}$ of the isometry group $\mathrm{Aut}(L_\mathfrak{g})$ generated by the reflections with respect to the hyperplanes perpendicular to roots is called the *Weyl group* of the Lie algebra $\mathfrak{g}$. The entire isometry group $\mathrm{Aut}(L_\mathfrak{g})$ is known to be the semidirect product $W_\mathfrak{g} : \mathrm{Aut}(\text{Dynkin diagram})$.

The Weyl group W_G is also defined for a connected compact Lie group G as follows. Let $T_G \cong \mathrm{U}(1)^r$ be the maximal torus in G, and $N(T_G) := \{g \in G \mid gtg^{-1} \in T_G \text{ for any } t \in T_G\}$ be its normalizer. Since the conjugate action of T_G on T_G is trivial, T_G is a normal subgroup of $N(T_G)$. The *Weyl group* of the Lie group G is then $W_G := N(T_G)/T_G$. It is known that W_G coincides with the Weyl group $W_\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of the Lie group G. The rank of $L_\mathfrak{g}$ is r.

Whether $N(T_G) = T_G.W_G \cong \mathrm{U}(1)^r.W_\mathfrak{g}$ splits or not, which is an analogous problem to the main question of [Oka24], is answered in [CWW74, Theorem 2] for simple Lie groups. It depends on the Lie group G, and for example, it does not split when $G = E_6, E_7, E_8$.

classified as the A_n lattices ($n \in \mathbb{Z}_{\geq 1}$), the D_n lattices ($n \in \mathbb{Z}_{\geq 4}$), and the E_6, E_7, E_8 lattices, defined as follows [CS99, Ch. 4].

- The A_n *lattice* is a root lattice of rank n defined as

$$A_n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\}, \tag{2.15}$$

on the hyperplane $\mathbb{R}^n \cong \{x_1 + \cdots + x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$, with the standard Euclidean metric on $\mathbb{R}^{n+1}$.

We can take a $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of A_n as

$$e_i = (0, \ldots, \overset{i}{\check{1}}, -1, \ldots, 0) \in \mathbb{R}^{n+1}, \tag{2.16}$$

and then the Gram matrix is

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{2.17}$$

- The D_n *lattice* is a root lattice of rank n defined as

$$D_n := \{k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \mid \sum_{i=1}^{n} k_i \equiv 0 \bmod 2\}, \tag{2.18}$$

with the standard Euclidean metric on $\mathbb{R}^n$.

We can take a $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of D_n as

$$e_1 = (-1, -1, 0, \ldots, 0), \tag{2.19}$$

$$e_i = (0, \ldots, \overset{i-1}{\check{1}}, -1, \ldots, 0) \quad (i = 2, \ldots, n), \tag{2.20}$$

and then the Gram matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{2.21}$$

In addition, the D_n^+ *lattice* is defined as

$$D_n^+ := D_n \sqcup (\frac{1}{2}\vec{1} + D_n). \tag{2.22}$$

- The E_8 *lattice* is a root lattice of rank 8 defined as

$$E_8 := \{k = (k_1, \ldots, k_8) \in \mathbb{Z}^8 \sqcup (\mathbb{Z} + \frac{1}{2})^8 \mid \sum_{i=1}^{n} k_i \equiv 0 \bmod 2\}, \tag{2.23}$$

with the standard Euclidean metric on $\mathbb{R}^n$. We have $D_8^+ = E_8$.

We can take a $\mathbb{Z}$-basis $e_1, \ldots, e_8$ of E_8 as

$$e_1 = (-1, -1, 0, \ldots, 0), \tag{2.24}$$
$$e_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \tag{2.25}$$
$$e_i = (0, \ldots, \overset{i-1}{\check{1}}, -1, \ldots, 0) \quad (i = 3, \ldots, 8), \tag{2.26}$$

and then the Gram matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & -1 & \cdots & 0 & 0 \\ -1 & 0 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{2.27}$$

The E_7 *lattice* and the E_6 *lattice* are defined as

$$E_7 := \{(k_1, \ldots, k_8) \in E_8 \mid k_7 = k_8\}, \tag{2.28}$$
$$E_6 := \{(k_1, \ldots, k_8) \in E_8 \mid k_6 = k_7 = k_8\}. \tag{2.29}$$

The Niemeier lattices can be distinguished by their root sublattices, that is, the sublattice generated by its vectors with squared length 2. A Niemeier lattice with its root sublattice being X is often referred to as the X Niemeier lattice, say the $(A_1)^{24}$ Niemeier lattice, except for the Leech lattice Λ_{24} whose root sublattice is 0.

Lattices from codes

There are several ways to construct a lattice from a given code. We focus on the case of binary codes here. See [CS99] for more on constructions.

Let $\mathcal{C} \subset (\mathbb{F}_2)^n$ be a binary code of length n, and regard each codeword of $\mathcal{C}$ as a vector with entries 0 or 1. We can construct a lattice $\Lambda(\mathcal{C})$ of rank n as

$$\Lambda(\mathcal{C}) := \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}^n \subset \mathbb{R}^n, \tag{2.30}$$

with the standard Euclidean metric $k \cdot k' = \sum_i k_i k_i'$ for $k = (k_i)_i, k' = (k_i')_i \in \mathbb{R}^n$ as the symmetric bilinear form. This construction is called Construction A [CS99, Ch. 7 §2]. This lattice $\Lambda(\mathcal{C})$ satisfies $\Lambda(\mathcal{C}^*) = \Lambda(\mathcal{C})^*$. Hence, $\Lambda(\mathcal{C})$ is integer if and only if $\mathcal{C}$ satisfies $\mathcal{C} \subset \mathcal{C}^*$, and self-dual if and only if $\mathcal{C}$ is self-dual. In addition, $\Lambda(\mathcal{C})$ is of Type I (odd self-dual) if and only if $\mathcal{C}$ is of Type I (singly-even self-dual), and Type II (even self-dual) if and only if $\mathcal{C}$ is of Type II (doubly-even self-dual). Another construction called Construction B [CS99, Ch. 7 §5] associates the sublattice

$$\Lambda_{\mathrm{B}}(\mathcal{C}) := \{k = (k_1, \ldots, k_n) \in \Lambda(\mathcal{C}) \mid \sqrt{2}\sum_{i=1}^{n} k_i \in 4\mathbb{Z}\} \tag{2.31}$$

of $\Lambda(\mathcal{C})$ to a binary code $\mathcal{C}$.

For a doubly-even self-dual binary code $\mathcal{C}$, we further consider the following constructions [DGM94, §5.1]. We first define

$$\mathbb{Z}^n_+ := \{x \in \mathbb{Z}^n \mid |x|^2 \in 2\mathbb{Z}\}, \tag{2.32}$$

$$\mathbb{Z}^n_- := \{x \in \mathbb{Z}^n \mid |x|^2 \in 2\mathbb{Z} + 1\}. \tag{2.33}$$

Recall that the length n of a doubly-even self-dual code is always a multiple of 8, and also define

$$\Lambda_0(\mathcal{C}) := \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}^n_+, \tag{2.34}$$

$$\Lambda_1(\mathcal{C}) := \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}^n_-, \tag{2.35}$$

$$\Lambda_2(\mathcal{C}) := \frac{1}{2\sqrt{2}}\vec{1} + \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}^n_{(-)^{\frac{n}{8}+1}}, \tag{2.36}$$

$$\Lambda_3(\mathcal{C}) := \frac{1}{2\sqrt{2}}\vec{1} + \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}^n_{(-)^{\frac{n}{8}}}, \tag{2.37}$$

where $\vec{1} := (1, \ldots, 1)$. All the vectors in $\Lambda_i(\mathcal{C})$ for $i = 0, 1, 3$ are even, whereas those in $\Lambda_2(\mathcal{C})$ are odd. Then we can construct some more lattices as in Table 2.2.

Table 2.2: Lattices constructed from a doubly-even self-dual binary code $\mathcal{C}$. On the right side of the vertical line, the names of lattices constructed from the binary Golay code G_{24} and their isometry groups are shown.

name of construction	lattice	property	for $\mathcal{C} = G_{24}$	Aut(lattice)
Construction A	$\Lambda(\mathcal{C}) = \Lambda_0(\mathcal{C}) \cup \Lambda_1(\mathcal{C})$	even self-dual	$(A_1)^{24}$ Niemeier lattice	$2^{24} : M_{24}$
Construction B	$\Lambda_{\mathrm{B}}(\mathcal{C}) = \Lambda_0(\mathcal{C})$	even		
twisted construction	$\tilde{\Lambda}(\mathcal{C}) = \Lambda_0(\mathcal{C}) \cup \Lambda_3(\mathcal{C})$	even self-dual	Leech lattice Λ_{24}	Co_0
–	$\tilde{\Lambda}'(\mathcal{C}) = \Lambda_0(\mathcal{C}) \cup \Lambda_2(\mathcal{C})$	odd self-dual	odd Leech lattice O_{24}	$2^{12} : M_{24}$

The odd Leech lattice O_{24}

The *odd Leech lattice* O_{24} is the unique positive-definite odd self-dual lattice of rank 24 without roots (vectors of squared length 2) up to isometry. It can be constructed from the binary Golay code G_{24} as

$$O_{24} = (\frac{1}{\sqrt{2}}G_{24} + \sqrt{2}\mathbb{Z}_+^{24}) \cup (\frac{1}{2\sqrt{2}}\vec{1} + \frac{1}{\sqrt{2}}G_{24} + \sqrt{2}\mathbb{Z}_+^{24}). \tag{2.38}$$

Historically, the odd Leech lattice was first found in [OP44].

If we use the binary Golay code G_{24} with the basis (2.6), then we can take a $\mathbb{Z}$-basis $e_1, \ldots, e_{24}$ of (2.38) as

$$\begin{aligned}
e_1 &= \tfrac{1}{2\sqrt{2}}\vec{1}, \\
e_2 &= \tfrac{1}{\sqrt{2}}(\text{the second line from the bottom of (2.6)}), \\
&\vdots \\
e_{12} &= \tfrac{1}{\sqrt{2}}(\text{the first line of (2.6)}), \\
e_{13} &= \sqrt{2}(1,0,0,0,0,0,0,0,0,0,0,1,0,\ldots,0), \\
e_{14} &= \sqrt{2}(1,0,0,0,0,0,0,0,0,0,1,0,0,\ldots,0), \\
&\vdots \\
e_{23} &= \sqrt{2}(1,1,0,\ldots,0), \\
e_{24} &= \sqrt{2}(2,0,\ldots,0).
\end{aligned} \tag{2.39}$$

In fact, O_{24} in (2.38) obviously contains $\mathrm{Span}_{\mathbb{Z}}\{e_i\}_i$, and the opposite direction of the inclusion can be checked as follows. Since it is obvious that $\mathrm{Span}_{\mathbb{Z}}\{e_i\}_i$ contains $\frac{1}{2\sqrt{2}}\vec{1}$ and $\frac{1}{\sqrt{2}}G_{24}$, it suffices to check that it also contains $\sqrt{2}\mathbb{Z}_+^{24}$. Here, $\mathbb{Z}_+^{24}$ is generated by the 24 vectors

$$\begin{aligned}
&(1,0,\ldots,0,0,1), \\
&(1,0,\ldots,0,1,0), \\
&\vdots \\
&(1,1,0,\ldots,0), \\
&(2,0,\ldots,0).
\end{aligned}$$

Since the last 12 vectors of $\sqrt{2}\times$(above 24 vectors) are exactly $e_{13}, \ldots, e_{24}$, it suffices to check that the first 12 vectors of $\sqrt{2}\times$(above 24 vectors) can be written as $\mathbb{Z}$-linear combinations of $e_1, \ldots, e_{24}$, which can be checked by computer.

The isometry group $\mathrm{Aut}(O_{24})$ of the odd Leech lattice is known to be $2^{12} : M_{24}$ [CS99, Ch. 17]. In the construction (2.38), these automorphisms are apparent because $M_{24} = \mathrm{Aut}(G_{24})$ and 2^{12} are the maps $k = (k_i)_i \mapsto ((-1)^{w_i}k_i)_i$ where $w = (w_i)_i \in G_{24}$.

If we use Construction A for ternary codes, an odd Leech lattice can also be constructed from any self-dual ternary code of length 24 with the minimal nonzero weight 9. It is known that there are only two such ternary codes up to equivalence [LPS81]: the extended quadratic residue code Q_{24} (see e.g. [MS77, Ch. 16]), and the symmetric code P_{24} defined by Pless [Ple69, Ple72] [MS77, Ch. 16 §8]. However, the automorphism groups of these ternary codes are $\mathrm{Aut}(Q_{24}) = (\mathbb{F}_3)^\times \times \mathrm{PSL}(23, \mathbb{F}_2)$ [MS77, Apply Theorem 13 in Ch. 16 §5 to $p = 23$] and $\mathrm{Aut}(P_{24}) = \mathbb{Z}_4 \times \mathrm{PGL}(11, \mathbb{F}_2)$ [MPS76, §5.2], so the structure of $\mathrm{Aut}(O_{24}) = 2^{12} : M_{24}$ is not apparent in these constructions of odd Leech lattices. See [GJF18, Example 4.5] for application of these constructions to $\mathcal{N} = 1$ supersymmetries of the lattice VOAs.

The Leech lattice Λ_{24}

The *Leech lattice* Λ_{24} is the unique positive-definite even self-dual lattice of rank 24 without roots (vectors of squared length 2) up to isometry. It can be constructed from the binary Golay code G_{24} as

$$\Lambda_{24} = (\frac{1}{\sqrt{2}}G_{24} + \sqrt{2}\mathbb{Z}^{24}_{+}) \cup (\frac{1}{2\sqrt{2}}\vec{1} + \frac{1}{\sqrt{2}}G_{24} + \sqrt{2}\mathbb{Z}^{24}_{-}). \tag{2.40}$$

Historically, the Leech lattice was discovered by Leech in [Lee67, §2.31], but it is also said that one of the more than 10 Niemeier lattices, which was reported to be found by Witt in [Wit41, p. 324] without further details, is the Leech lattice [Wit98, p. 328-329]. See an informative video [Bor20] by Borcherds for more details.

For an even lattice L, we define

$$L_d := \{k \in L \mid |k|^2 = 2d\}. \tag{2.41}$$

An even lattice L of rank 24 is isomorphic to the Leech lattice Λ_{24}, if and only if it satisfies

$$|L_1| = 0, \tag{2.42}$$
$$|L_2| = 196560, \tag{2.43}$$
$$|L_3| = 16773120, \tag{2.44}$$
$$|L_4| = 398034000. \tag{2.45}$$

We consider the quotient $\Lambda_{24}/2\Lambda_{24}$ of Λ_{24} by the equivalence relation $k \sim k' \Leftrightarrow k - k' \in 2\Lambda_{24}$. We can take a complete set of representatives of $\Lambda_{24}/2\Lambda_{24}$ as follows.

- Take $0 \in \Lambda_{24}$.
- From $(\Lambda_{24})_2$ and $(\Lambda_{24})_3$, if we take k, then do not take $-k$.
- For $k \in (\Lambda_{24})_4$, its equivalence class $k + 2\Lambda_{24}$ contains 48 vectors of squared length 8 as $(k + 2\Lambda_{24}) \cap (\Lambda_{24})_4 = \{\pm k, \pm k_2, \dots, \pm k_{24}\}$, so take one of them. It is known that $\{k, k_2, \dots, k_{24}\}$ constitutes an orthogonal basis of $\mathbb{R}^{24}$.

2.3.2 The Conway Groups

The Conway groups Co_0 and Co_1

The isometry group $\mathrm{Aut}(\Lambda_{24})$ of the Leech lattice is the largest *Conway group* Co_0. It is not a simple group, but its quotient by the center $\{\pm 1\}$ is the largest sporadic Conway group Co_1.

From the construction (2.40), it is apparent that Co_0 contains $2^{12} : M_{24}$ as a subgroup, where $M_{24} = \mathrm{Aut}(G_{24})$ and 2^{12} are the maps $k = (k_i)_i \mapsto ((-1)^{w_i} k_i)_i$ where $w = (w_i)_i \in G_{24}$. This subgroup $2^{12} : M_{24}$ is called the *monomial subgroup* of Co_0. Co_0 is generated by this monomial subgroup and a specific order-2 element. As a subgroup of $\mathrm{O}(\mathbb{R}^{24})$, Co_0 does not contain matrices of determinant -1, and hence Co_0 is a subgroup of $\mathrm{SO}(\mathbb{R}^{24})$.

Generators of Co_0 as a subgroup of $\mathrm{SL}(24, \mathbb{Z})$ can be found on the Co_1 page of [WWT+] as

$$A = \begin{pmatrix}
2 & 0 & 0 & -3 & -2 & -1 & 0 & -2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-4 & 1 & 0 & 6 & 4 & 2 & 2 & 3 & -1 & -3 & -1 & 0 & -2 & -1 & -1 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
4 & -1 & 0 & -6 & -4 & -2 & -3 & -4 & 2 & 4 & 3 & -3 & 2 & 1 & 2 & 0 & 1 & -1 & -1 & -1 & 1 & 0 & -1 & 1 \\
4 & 1 & 1 & -6 & -3 & -2 & -1 & -4 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 \\
5 & -2 & 0 & -8 & -5 & -3 & -2 & -4 & 2 & 4 & 2 & 0 & 2 & 2 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\
-3 & 0 & -1 & 4 & 1 & 2 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
1 & 0 & -1 & -2 & -2 & -1 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-3 & 1 & -1 & 5 & 3 & 1 & 2 & 2 & -1 & -2 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & 0 & 3 & 3 & 0 & 2 & 2 & 0 & -1 & -2 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \\
9 & -1 & -1 & -13 & -9 & -5 & -4 & -9 & 4 & 6 & 4 & -1 & 3 & 1 & 2 & 2 & -1 & -3 & -1 & -1 & -2 & 1 & -1 & 1 \\
1 & -1 & -1 & -3 & -3 & -1 & -1 & -4 & 2 & 2 & 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 \\
4 & -1 & 1 & -6 & -4 & -2 & -2 & -3 & 2 & 3 & 2 & -1 & 2 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 \\
-4 & 0 & 0 & 5 & 3 & 2 & 2 & 3 & -2 & -2 & -1 & 0 & -2 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 \\
8 & 2 & 2 & -9 & -5 & -4 & 0 & -5 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & -1 & 0 & -1 & 1 & -1 & 1 & 0 & 1 \\
-3 & 0 & 0 & 3 & 3 & 1 & 0 & 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\
-5 & 2 & 0 & 6 & 4 & 3 & 3 & 2 & -1 & -4 & -2 & 1 & -2 & -2 & -2 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\
3 & 0 & 0 & -3 & -3 & -1 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & -1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\
-4 & -1 & -1 & 4 & 2 & 2 & 0 & 2 & -1 & -2 & -1 & 1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\
-9 & 1 & 0 & 11 & 7 & 5 & 3 & 6 & -2 & -5 & -3 & 1 & -2 & -2 & -2 & -1 & 1 & 2 & 0 & 1 & 0 & -1 & 1 & 0 \\
-6 & 1 & 0 & 9 & 5 & 4 & 3 & 4 & -2 & -4 & -1 & -1 & -2 & -1 & -1 & -2 & 1 & 2 & 0 & 0 & 1 & -1 & 0 & 0 \\
5 & 1 & -1 & -7 & -5 & -3 & -1 & -6 & 2 & 3 & 3 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 \\
6 & 0 & 1 & -6 & -4 & -2 & -2 & -3 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}^T, \tag{2.46}$$

$$B = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
3 & 1 & -1 & -3 & -2 & -2 & 0 & -3 & 1 & 2 & 2 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 \\
-4 & 0 & -1 & 7 & 4 & 3 & 1 & 4 & -3 & -3 & -2 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & -1 \\
0 & -1 & 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 2 & -6 & -4 & -2 & -3 & -2 & 1 & 2 & 1 & -1 & 1 & 1 & 2 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \\
2 & 0 & -1 & -5 & -3 & -3 & -1 & -4 & 3 & 3 & 1 & 0 & 2 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 1 \\
-10 & 0 & 0 & 12 & 8 & 5 & 4 & 7 & -3 & -6 & -3 & 1 & -3 & -1 & -3 & -1 & 0 & 3 & 1 & 1 & 1 & -1 & 1 & -1 \\
-1 & 4 & 1 & 5 & 4 & 1 & 3 & 3 & -2 & -3 & -2 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 \\
-1 & 0 & 1 & 3 & 2 & 1 & -1 & 3 & -1 & -1 & -2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-5 & 3 & 0 & 11 & 7 & 4 & 4 & 6 & -4 & -5 & -3 & 0 & -2 & -2 & -1 & -2 & 1 & 1 & 1 & -1 & 1 & -2 & 1 & -1 \\
3 & 0 & 1 & -2 & -1 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-4 & 0 & -1 & 5 & 3 & 2 & 2 & 2 & -1 & -3 & 0 & 0 & -2 & -1 & -2 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\
2 & -1 & 1 & -2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\
2 & -1 & 1 & -3 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\
-2 & 0 & -1 & 3 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-11 & -2 & -1 & 12 & 7 & 5 & 3 & 7 & -2 & -5 & -3 & 1 & -2 & -1 & -3 & -2 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\
-2 & 0 & 1 & 2 & 2 & 1 & 0 & 3 & -1 & -1 & -2 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-6 & 1 & -1 & 7 & 5 & 2 & 2 & 4 & -2 & -2 & -3 & 1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
-3 & 1 & -1 & 3 & 2 & 1 & 2 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
2 & -2 & -2 & -6 & -5 & -2 & -2 & -6 & 3 & 4 & 3 & -1 & 2 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 \\
2 & -1 & -2 & -4 & -4 & -1 & -1 & -5 & 2 & 2 & 4 & -2 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -1 & -1 & 0 \\
-6 & 1 & 0 & 8 & 5 & 4 & 3 & 3 & -2 & -4 & -1 & 0 & -2 & -1 & -2 & -1 & 0 & 2 & 0 & 0 & 0 & -2 & 1 & -1 \\
14 & 1 & 2 & -18 & -11 & -7 & -4 & -10 & 4 & 7 & 4 & 0 & 3 & 2 & 3 & 3 & -1 & -2 & -2 & 0 & -1 & 2 & -1 & 2
\end{pmatrix}^T, \tag{2.47}$$

where the transpose on the right-hand side is taken just for the convenience of notation. These generators satisfy $A^2 = -1$ and $B^3 = 1$.

The Conway group Co_0 generated by these generators A and B is the isometry group of a lattice $(\mathbb{Z}^{24}, q)$, where q is some symmetric bilinear form which makes $(\mathbb{Z}^{24}, q)$ isomorphic to the Leech lattice. The matrix form of q up to scalar multiplication can be found by solving the conditions

$$A^T q A = q, \quad B^T q B = q, \tag{2.48}$$

by Mathematica [Wol23]. Note that the elements of the Conway group Co_0 in this notation act on the column vectors in the lattice $(\mathbb{Z}^{24}, q)$ from the left. The result is

$$q = \begin{pmatrix}
4 & -2 & -2 & 2 & 2 & 2 & -1 & -1 & 2 & 2 & -2 & 2 & -2 & 1 & -1 & 1 & 1 & 2 & 0 & -1 & 0 & -2 & -2 & 2 \\
-2 & 4 & 0 & 0 & -2 & 0 & -1 & 2 & 0 & 0 & 2 & -1 & 1 & 1 & -1 & 1 & -2 & -2 & -1 & 1 & 1 & 2 & 1 & -2 \\
-2 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 0 & -1 \\
2 & 0 & 0 & 4 & 0 & 0 & -2 & -1 & 2 & 2 & 0 & 2 & -2 & 1 & 0 & 2 & -1 & 0 & -1 & 0 & 1 & -1 & -1 & 0 \\
2 & -2 & 0 & 0 & 4 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 2 & 0 & -2 & -2 & -1 & -2 & 2 \\
2 & 0 & -2 & 0 & 0 & 4 & 0 & 0 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 1 \\
-1 & -1 & 0 & -2 & -1 & 0 & 4 & 1 & -1 & -1 & 0 & -2 & 2 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & -1 & 1 & 2 & -1 \\
-1 & 2 & 0 & -1 & -1 & 0 & 1 & 4 & 1 & 1 & 2 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\
2 & 0 & 0 & 2 & 1 & 1 & -1 & 1 & 4 & 2 & 0 & 2 & -2 & 1 & 0 & 1 & -1 & 0 & -1 & -2 & 0 & -1 & -1 & 0 \\
2 & 0 & 0 & 2 & 1 & 1 & -1 & 1 & 2 & 4 & 0 & 1 & -2 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & -1 & -1 & -1 & 1 \\
-2 & 2 & 2 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 4 & 0 & 1 & -1 & 0 & 1 & -1 & -2 & 0 & 0 & 0 & 2 & 1 & -2 \\
2 & -1 & 0 & 2 & 1 & 1 & -2 & -1 & 2 & 1 & 0 & 4 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & -2 & -2 & 1 \\
-2 & 1 & 0 & -2 & -1 & -1 & 2 & 1 & -2 & -2 & 1 & -2 & 4 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & 0 & 2 & 2 & -2 \\
1 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 4 & -1 & 1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 1 & -1 & 2 & -1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & -1 & 4 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 \\
1 & -2 & -1 & -1 & 0 & 1 & 2 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -1 & 0 & 2 & 0 & -1 & -1 & 0 & 1 & -2 & 1 & -1 & -1 & 0 & -1 & 1 & 4 & 1 & -1 & -1 & -2 & -2 & 2 \\
0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -2 & 0 & -1 & 1 & 1 & 4 & 0 & 0 & 0 & -1 & 1 \\
-1 & 1 & -1 & 0 & -2 & 0 & 0 & -1 & -2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & 2 & 0 & 2 & -1 \\
0 & 1 & -1 & 1 & -2 & 1 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 2 & 4 & 0 & 1 & -1 \\
-2 & 2 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & -1 & 2 & -2 & 2 & 0 & -1 & 1 & 0 & -2 & 0 & 0 & 0 & 4 & 1 & -2 \\
-2 & 1 & 0 & -1 & -2 & -1 & 2 & 1 & -1 & -1 & 1 & -2 & 2 & 0 & 0 & 0 & 0 & -2 & -1 & 2 & 1 & 1 & 4 & -2 \\
2 & -2 & -1 & 0 & 2 & 1 & -1 & -1 & 0 & 1 & -2 & 1 & -2 & 0 & 0 & -1 & 0 & 2 & 1 & -1 & -1 & -2 & -2 & 4
\end{pmatrix}. \tag{2.49}$$

The Conway groups Co_2 and Co_3

There are two more sporadic simple Conway groups Co_2 and Co_3. They are the stabilizer groups of one vector in $(\Lambda_{24})_2$ and $(\Lambda_{24})_3$, respectively, under the action of Co_0 on Λ_{24}. It is known that the actions of Co_0 on $(\Lambda_{24})_2$ and $(\Lambda_{24})_3$ are transitive, so these groups are unique up to isomorphism. Since $-1 \in \mathrm{Co}_0$ does not preserve any non-zero vector, Co_2 and Co_3 are also isomorphic to subgroups of Co_1.

2.4 The Monster Group

The monster group $\mathbb{M}$, whose existence was predicted independently by Fischer and Griess in 1973, was first constructed by Griess in 1981 [Gri81, Gri82], as the automorphism group of a certain algebra $B^\natural$ with a bilinear form τ. The algebra $B^\natural$ is called the Griess algebra. More precisely, Griess constructed $\mathbb{M}$ as a subgroup of $\mathrm{Aut}(B^\natural, \tau)$, and $\mathrm{Aut}(B^\natural, \tau) = \mathbb{M}$ was later shown in [Tit84].

As a vector space, $B^\natural$ is a 196884-dimensional representation space of a group $C(\Lambda_{24}) = 2^{1+24}.\mathrm{Co}_1$ over $\mathbb{Q}$. In [Gri82], a specific commutative non-associative product and an associative[6] symmetric bilinear form τ are introduced to this $C(\Lambda_{24})$-module, in a way compatible with the

[6] A bilinear form τ is called *associative* if it satisfies $\tau(u, v \cdot w) = \tau(u \cdot v, w)$.

$C(\Lambda_{24})$-action, which defines the $C(\Lambda_{24})$-algebra $B^\natural$ called the *Griess algebra*. Now, $\mathrm{Aut}(B^\natural, \tau)$ contains $C(\Lambda_{24})$, and a specific order-2 element σ can be found in $\mathrm{Aut}(B^\natural, \tau) \setminus C(\Lambda_{24})$. Finally, the group generated by $C(\Lambda_{24})$ and σ is shown to be simple, and have the right order $|\mathbb{M}|$.

When $B^\natural$ was introduced in [Gri82], the monster VOA $V^\natural$ [FLM88] (explained later in Section 5) was not yet constructed. However, in the modern understanding, it is natural to regard $B^\natural$ as the weight-2 subspace $(V^\natural)_2$ of $V^\natural$. In fact, one of the important results on the monstrous moonshine is that $\mathrm{Aut}(V^\natural) = \mathbb{M}$. As we will see in Section 5, $V^\natural$ is the $\mathbb{Z}_2$-orbifold of the Leech lattice VOA, and hence its weight-2 subspace $(V^\natural)_2 = B^\natural$ consists of

- $\mathrm{Span}\{\alpha^i_{-1}\alpha^j_{-1}|0\rangle\}_{i,j=1,\ldots,24}$, which is the symmetric tensor product of $\mathbf{24}$, so the dimension is $\frac{25\times 24}{2} = 1 + 299$, containing the trivial representation $\mathrm{Span}\{\sum_i \alpha^i_{-1}\alpha^i_{-1}|0\rangle\}$ of $\mathrm{SO}(24)$.
- $\mathrm{Span}\{|k\rangle + |-k\rangle\}_{k\in(\Lambda_{24})_2}$, whose dimension is $|(\Lambda_{24})_2|/2 = 98280$.
- $\mathrm{Span}\{c^i_{-\frac{1}{2}}|s\rangle\}_{i=1,\ldots,24,|s\rangle\in\mathcal{X}(\Lambda_{24})}$, where $\mathcal{X}(\Lambda_{24})$ is the $2^{\frac{24}{2}}$-dimensional irreducible representation of a certain gamma matrix algebra $\Gamma(\Lambda_{24})$, so the dimension is $24 \times 2^{\frac{24}{2}} = 98304$.

These three subspaces are denoted by U, V, and W, respectively, in [Gri82]. In the language of VOA (see Section 4), the commutative non-associative product on $B^\natural$ is introduced as

$$v \cdot w = v_{(1)}w, \tag{2.50}$$

and the associative symmetric bilinear form τ is introduced as

$$\tau(v, w)\mathbf{1} = v_{(3)}w, \tag{2.51}$$

for any $v, w \in (V^\natural)_2$. The one-dimensional subspace in the first $(1 + 299)$-dimensional subspace will turn out to be the trivial representation of $\mathbb{M} = \mathrm{Aut}(B^\natural, \tau)$, and the irreducible decomposition of $B^\natural$ as a representation of $\mathbb{M}$ is $196884 = 1 + 196883$, where 196883 is the smallest nontrivial irreducible representation of $\mathbb{M}$.

See for example [Gri98, Ch. 11] and [Har99] for more on the Griess algebra and the monster group. A simpler construction of the monster group was provided in [Con85], and reviewed in [CS99, Ch. 29]. See [Iva09] for another construction using the amalgam method.

3 Modular Functions and Weak Jacobi Forms

This Section 3 collects the basic definitions and important examples of modular functions and weak Jacobi forms. After mentioning the action of the modular group on the upper half-plane in Section 3.1, we will review modular functions and weak Jacobi forms in Sections 3.2 and 3.3, respectively. In the last Section 3.4, we will review the elliptic genes of CFT, through which weak Jacobi forms appear in physics.

More details on the facts cited here about the modular functions can be found, for example, in [Ser73, Ch. VII] and [DS05, Ch. 1]. The foundational literature on weak Jacobi forms is [EZ85], and we will follow a summary in [DMZ12, §4].

3.1 Modular Group

The special linear group

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\} \tag{3.1}$$

is generated by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

It acts on the upper half-plane of the complex plane

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \tag{3.3}$$

as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau+b}{c\tau+d}\,. \tag{3.4}$$

If we define a complex torus as $E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, then it is known that[7] E_τ and $E_{\tau'}$ are isomorphic as complex manifolds if and only if there is $g \in \mathrm{SL}(2,\mathbb{Z})$ such that $g \cdot \tau = \tau'$.

Since the center $\mathbb{Z}_2 = \langle -I \rangle$ of $\mathrm{SL}(2,\mathbb{Z})$ acts trivially on $\mathbb{H}$, we can say that $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\langle -I \rangle$ acts on $\mathbb{H}$. This group $\mathrm{PSL}(2,\mathbb{Z})$ is called the *modular group*. However, in the following discussions, it suffices to use $\mathrm{SL}(2,\mathbb{Z})$. In fact, some literature also calls $\mathrm{SL}(2,\mathbb{Z})$ the modular group.

Lastly, we remark that $\mathrm{GL}(2,\mathbb{Z})$ is generated by T, S, and

$$P := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

[7] This fact can be understood as follows [ES15, §5.1]. The lattice in the complex plane spanned by $\omega_1, \omega_2 \in \mathbb{C}$ is the same as the one spanned by $\omega_1', \omega_2' \in \mathbb{C}$ such that $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. (A general $\mathrm{GL}(2,\mathbb{Z})$ transformation may reverse the orientation of the lattice.) In addition, a torus $\mathbb{C}/(\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z})$ is isomorphic to $\mathbb{C}/(\lambda\omega_1\mathbb{Z} \oplus \lambda\omega_2\mathbb{Z})$ where $\lambda \in \mathbb{C}$, so we can regard the $\mathrm{SL}(2,\mathbb{Z})$ transformations on ω_1, ω_2 as those on $\tau := \frac{\omega_2}{\omega_1}$.

3.2 Modular Functions

Definition 3.1 (modular function and modular form)**.**

(1) A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *weakly modular function* of *weight* $k \in \mathbb{Z}$ if it satisfies

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau) \quad \text{for any} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \tag{3.6}$$

or equivalently,

$$f(\tau + 1) = f(\tau), \quad f(-\frac{1}{\tau}) = \tau^k f(\tau). \tag{3.7}$$

From this periodicity under $\tau \mapsto \tau + 1$, we can regard f as a function of $q := e^{2\pi\sqrt{-1}\tau}$.

(2) A weakly modular function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular function* if, as a function of q, it extends to a meromorphic function at $q = 0$. In such a case, we will simply say that f is meromorphic at $q = 0$ or[8] at $\tau = \infty$.

By this definition, a modular function f admits a Laurent expansion at $q = 0$,

$$f(\tau) = \sum_{n=n_0}^{\infty} c_n q^n. \tag{3.8}$$

(3) A modular function $f : \mathbb{H} \to \mathbb{C}$ is called a *weakly holomorphic modular form* if it is holomorphic on $\mathbb{H}$ but not necessarily at $q = 0$. That is, a pole is allowed to exist only at $q = 0$.

(4) A weakly holomorphic modular form $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form* if it is holomorphic at $q = 0$. In this case, $f(\infty)$ denotes the value of (the extended) $f(\tau)$ at $q = 0$.

(5) A modular form $f : \mathbb{H} \to \mathbb{C}$ is called a *cusp form* if $f(\infty) = 0$.

■

There is no non-zero weakly modular function of odd weight. This is because, if k is odd, we have $f(\tau) = -f(\tau)$ by applying $a = d = -1$ and $b = c = 0$ to (3.6).

Here are some examples of modular functions and modular forms.

[8] Here, ∞ denotes the point at infinity. Of course, as a limit, $q \to 0$ as $\mathrm{Im}(\tau) \to \infty$.

- Let $k \geq 4$ be an even integer. The *Eisenstein series* $G_k(\tau)$ of weight k defined as follows is a modular form of weight k.

$$G_k(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^k} \tag{3.9}$$

$$= 2\zeta(k) + 2\frac{(2\pi\sqrt{-1})^k}{(k-1)!}\sum_{c=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}q^{cd} \tag{3.10}$$

$$= 2\zeta(k) + 2\frac{(2\pi\sqrt{-1})^k}{(k-1)!}\sum_{n=1}^{\infty}\left(\sum_{d|n} d^{k-1}\right)q^n. \tag{3.11}$$

Here, $\zeta(k) := \sum_{n=1}^{\infty}\frac{1}{n^k}$ is the Riemann zeta function, and its value for even k is known to be $\zeta(k) = -\frac{(2\pi\sqrt{-1})^k}{2k!}B_k$, where B_k is the Bernoulli number defined as $\frac{x}{e^x-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}x^k$. To calculate it, the recurrence relation $B_k = -\frac{1}{k+1}\sum_{i=0}^{k-1}\binom{k+1}{i}B_i$ and $B_0 = 1$ are useful.

The *normalized Eisenstein series*

$$E_k(\tau) := \frac{1}{2\zeta(k)}G_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\left(\sum_{d|n} d^{k-1}\right)q^n \tag{3.12}$$

is also frequently used. For example, since $B_4 = -\frac{1}{30}$ and $B_6 = \frac{1}{42}$,

$$E_4(\tau) = \frac{720}{(2\pi)^4}G_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\left(\sum_{d|n} d^3\right)q^n \tag{3.13}$$

$$= 1 + 240q + 2160q^2 + \cdots, \tag{3.14}$$

$$E_6(\tau) = \frac{30240}{(2\pi)^6}G_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\left(\sum_{d|n} d^5\right)q^n \tag{3.15}$$

$$= 1 - 504q - 16632q^2 - \cdots. \tag{3.16}$$

- The *modular discriminant* $\Delta(\tau)$ defined as follows is a cusp form of weight 12.

$$\Delta(\tau) := (60G_4(\tau))^3 - 27(140G_6(\tau))^2 \tag{3.17}$$

$$= \frac{(2\pi)^{12}}{1728}(E_4(\tau)^3 - E_6(\tau)^2) \tag{3.18}$$

$$= (2\pi)^{12}\eta(\tau)^{24} \tag{3.19}$$

$$= q\prod_{n=1}^{\infty}(1-q^n)^{24}, \tag{3.20}$$

where $\eta(\tau)$ is the Dedekind eta function defined in Appendix B as (B.38). The normalized one

$$\tilde{\Delta}(\tau) := \frac{1}{(2\pi)^{12}}\Delta(\tau) = \eta(\tau)^{24} \tag{3.21}$$

is also frequently used.

- The *modular j-function* or the *j-invariant* $j(\tau)$ defined as follows is a weakly holomorphic modular function of weight 0.

$$j(\tau) := 1728\frac{(60G_4(\tau))^3}{\Delta(\tau)} = \frac{E_4(\tau)^3}{\tilde{\Delta}(\tau)} \tag{3.22}$$

$$= q^{-1} + 744 + 196884q + 21493760q^2 + \cdots . \tag{3.23}$$

  The weight of $j(\tau)$ being 0 means that $j : \mathbb{H} \to \mathbb{C}$ is a function which is also well-defined on $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. In addition, $j : \mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}$ is known to be a bijection. As remarked in Section 3.1, elements of $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ are in one-to-one correspondence with the isomorphism classes of complex tori as $[\tau] \leftrightarrow [E_\tau]$. Therefore, $j(\tau)$ gives a complete invariant of the isomorphism classes of complex tori.

- Let L be a positive-definite lattice of rank n. The *theta function of the lattice* L is defined as

$$\Theta_L(\tau) := \sum_{k\in L} q^{\frac{1}{2}|k|^2}. \tag{3.24}$$

  If L is even self-dual, then $\Theta_L(\tau)$ is a modular form of weight $\frac{n}{2}$.

  For example, the theta function of the E_8 lattice is $\Theta_{E_8}(\tau) = E_4(\tau)$. We can see it as follows. The constant term of $\Theta_{E_8}(\tau)$ is of course $\Theta_{E_8}(\tau) = 1 + O(q)$. As in the following Theorem 3.2, the ring of modular forms is $\mathbb{C}[E_4, E_6]$, so this constant term determines the modular form $\Theta_{E_8}(\tau)$ of weight 4 as $\Theta_{E_8}(\tau) = E_4(\tau)$. In particular, we can say that

$$j(\tau) = \frac{\Theta_{E_8^{\oplus 3}}(\tau)}{\eta(\tau)^{24}}. \tag{3.25}$$

  More generally, we sometimes define the theta function $\Theta_L(\tau, \overline{\tau})$ of a lattice L of signature (r, s) (see (5.27)).

Obviously, all modular functions of the same weight constitute a $\mathbb{C}$-vector space. In addition, the multiplication of two modular functions of weight m and m' is a modular function of weight $m + m'$. Therefore, if we allow the sum of modular functions of inhomogeneous weights, then we obtain the graded ring of all (the inhomogeneous sums of) modular functions. Furthermore, its subrings are known to have the following structures. Let $R[X_1, \ldots, X_r]$ denote the polynomial ring over a commutative ring R.

Theorem 3.2.

- *The ring of modular functions of weight 0 constitutes the rational function field* $\mathbb{C}(j)$.
- *The ring of weakly holomorphic modular forms is* $\mathbb{C}[E_4, E_6, \tilde{\Delta}^{-1}]$.

  In particular, the ring of weakly holomorphic modular forms of weight 0 is $\mathbb{C}[j]$.

- *The ring of modular forms is* $\mathbb{C}[E_4, E_6]$.

  In particular, there is no non-zero modular form of weight $k < 4$.

- *The ring of weakly holomorphic modular forms with integral* q*-expansion coefficients* ($c_n \in \mathbb{Z}$ *in* (3.8)) *is*[9] $(\mathbb{Z}[E_4, E_6, \tilde{\Delta}]/\sim)[\tilde{\Delta}^{-1}]$, *where the relation* $\sim$ *is defined by* $1728\tilde{\Delta} \sim (E_4)^3 - (E_6)^2$.

- *The ring of modular forms with integral* q*-expansion coefficients* ($c_n \in \mathbb{Z}$ *in* (3.8)) *is* $\mathbb{Z}[E_4, E_6, \tilde{\Delta}]/\sim$, *with the above relation* $\sim$.

3.3 Weak Jacobi Forms

Definition 3.3. Let $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying

$$\varphi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi\sqrt{-1}m\frac{cz^2}{c\tau+d}}\varphi(\tau, z) \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \tag{3.26}$$

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi\sqrt{-1}m(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z) \qquad \text{for any } \lambda, \mu \in \mathbb{Z}, \tag{3.27}$$

where $k \in \mathbb{Z}$ is called the *weight* and $m \in \mathbb{Z}_{>0}$ is called the *index*. These transformations are equivalent to

$$\varphi(\tau+1, z) = \varphi(\tau, z), \qquad \varphi(-\frac{1}{\tau}, \frac{z}{\tau}) = \tau^k e^{2\pi\sqrt{-1}m\frac{z^2}{\tau}}\varphi(\tau, z), \tag{3.28}$$

$$\varphi(\tau, z+1) = \varphi(\tau, z), \qquad \varphi(\tau, z+\tau) = e^{-2\pi\sqrt{-1}m(\tau+2z)}\varphi(\tau, z). \tag{3.29}$$

(The last transformation $\varphi(\tau, z+\tau)$ also follows from those of $\varphi(-\frac{1}{\tau}, \frac{z}{\tau})$ and $\varphi(\tau, z+1)$.) From the periodicity under $\tau \mapsto \tau + 1$ and $z \mapsto z + 1$, we can regard φ as a function of $q := e^{2\pi\sqrt{-1}\tau}$ and $y := e^{2\pi\sqrt{-1}z}$. We further assume that φ admits a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r) q^n y^r. \tag{3.30}$$

(1) φ is called a *weakly holomorphic Jacobi form* if there exists $n_0 \in \mathbb{Z}$ such that $c(n, r) = 0$ for $n < n_0$.

(2) φ is called a *weak Jacobi form* if $c(n, r) = 0$ for $n < 0$.

(3) φ is called a *(holomorphic) Jacobi form* if $c(n, r) = 0$ for $4mn < r^2$.

(4) φ is called a *Jacobi cusp form* if $c(n, r) = 0$ for $4mn \le r^2$.

■

[9] See https://mathoverflow.net/questions/386020/modular-forms-over-mathbbz-vs-modular-forms-with-integral-fourier-coefficie.

If we set $m = 0$, then $\varphi(\tau, z)$ as a function of z is a holomorphic doubly-periodic function, and such a function is known to be constant with respect to z. Therefore, a weakly holomorphic Jacobi form of weight k and index 0 is just a modular function of weight k. We can also say that if $\varphi(\tau, z)$ is a weakly holomorphic Jacobi form of weight k and index m, then $\varphi(\tau, 0)$ is a modular function of weight k. Similarly to the case of modular functions, there is no non-zero weakly holomorphic Jacobi form of odd weight.

Important examples of weak Jacobi forms are[10]

$$\varphi_{-2,1}(\tau, z) := -\frac{\theta_1(\tau, z)^2}{\eta(\tau)^6}, \tag{3.31}$$

$$\varphi_{0,1}(\tau, z) := 4\left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right), \tag{3.32}$$

where $\theta_i(\tau, z)$ $(i = 1, 2, 3, 4)$ and $\eta(\tau)$ are the elliptic theta functions and the Dedekind eta function, respectively, introduced in Appendix B. These $\varphi_{k,m}(\tau, z)$ are weak Jacobi forms of weight k and index m.

Similarly to the case of modular functions, we have the graded ring of all (the inhomogeneous sums of) weak Jacobi forms. It has the following structure. If we write the $\mathbb{C}$-vector space of modular forms of weight k as M_k, then recall that the graded ring of modular forms is $\bigoplus_{k\in\mathbb{Z}} M_k = \mathbb{C}[E_4, E_6]$.

Theorem 3.4. *Let $J^{\text{weak}}_{k,m}$ denote the ring of weak modular forms of weight k and index m. The ring of weak Jacobi forms has the structure*

$$\bigoplus_{k\in\mathbb{Z}, m\in\mathbb{Z}_{>0}} J^{\text{weak}}_{k,m} = \left(\bigoplus_{k\in\mathbb{Z}} M_k\right)[\varphi_{-2,1}, \varphi_{0,1}] \tag{3.33}$$

$$= \mathbb{C}[E_4, E_6, \varphi_{-2,1}, \varphi_{0,1}]. \tag{3.34}$$

In particular,

$$J^{\text{weak}}_{k,m} = \bigoplus_{j=0}^{m} M_{k+2j} \cdot \varphi^{j}_{-2,1}\varphi^{m-j}_{0,1}. \tag{3.35}$$

Table 3.1: Generators of the ring of weak Jacobi forms.

	weight k	index m	Fourier expansion
E_4	4	0	$1 + 240q + 2160q^2 + \cdots$
E_6	6	0	$1 - 504q - 16632q^2 - \cdots$
$\varphi_{-2,1}$	-2	1	$y - 2 + y^{-1} - 2(y - 2 + y^{-1})^2 q + \cdots$
$\varphi_{0,1}$	0	1	$y + 10 + y^{-1} + (10y^2 - 64y + 108 - 64y^{-1} + 10y^{-2})q + \cdots$

[10] The minus sign in (3.31) is needed to obtain $\varphi_{-2,1}(\tau, z) = (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2 + \cdots$, because we have $\sqrt{-1}$ in the definition (B.21) of θ_1.

Extremal elliptic genus

As an example, the weak Jacobi form $Z_{\text{ext}}^{m=4}(\tau,z)$ of weight 0 and index 4, starting with the terms

$$Z_{\text{ext}}^{m=4}(\tau,z) = y^4 + 0(y^3+y^2+y)+\cdots \tag{3.36}$$

can be determined from (3.35) and Table 3.1 as

$$Z_{\text{ext}}^{m=4}(\tau,z) = \frac{1}{432}\varphi_{0,1}^4 + \frac{1}{8}E_4\varphi_{-2,1}^2\varphi_{0,1}^2 + \frac{11}{27}E_6\varphi_{-2,1}^3\varphi_{0,1} + \frac{67}{144}E_4^2\varphi_{-2,1}^4. \tag{3.37}$$

This is an example of the $\mathcal{N}=2$ extremal elliptic genus defined[11] by [GGK+08].

Definition 3.5 ($\mathcal{N}=2$ extremal elliptic genus)**.** We define the *polar region* of index $m\in\mathbb{Z}_{>0}$ as

$$\mathcal{P}^{(m)} := \{(n,r)\in\mathbb{Z}^2 \mid 0\le n, 4mn<r^2, 1\le r\le m\}, \tag{3.38}$$

and the terms q^ny^r of the Fourier expansion of a function $f(\tau,z)$ such that $(n,r)\in\mathcal{P}^{(m)}$ are called the *polar terms* of index m of f.

An $\mathcal{N}=2$ *extremal elliptic genus* $Z_{\text{ext}}^{m}(\tau,z)$ of index m or of central charge $6m$ is a weak Jacobi form of weight 0 and index m such that its polar terms of index m coincide with the polar terms of index m of the character of the Ramond vacuum representation[12] of the $\mathcal{N}=2$ superconformal algebra of central charge $c=6m$. Such polar terms can be explicitly calculated as the polar terms of

$$(-1)^m(1-q)y^m\prod_{l=1}^{\infty}\frac{(1-yq^{l+1})(1-y^{-1}q^l)}{(1-q^l)^2}. \tag{3.39}$$

■

According to [GGK+08], the number of polar terms of index m exceeds the number of $\mathbb{C}$-linearly independent terms of $J_{0,m}^{\text{weak}}$ in (3.35). Therefore, an $\mathcal{N}=2$ extremal elliptic genus $Z_{\text{ext}}^{m}(\tau,z)$ is unique if it exists. In addition, they showed that it exists only when $m=1,2,3,4,$ $5,7,8,11,13$ for $m\le 36$, and it does not exist for sufficiently large m. So they conjecture that it exists only for the listed values of m. The $\mathcal{N}=2$ extremal elliptic genera for $m=1,2,3,4$

[11] An *extremal VOA* is defined in [Hoe07] as a self-dual VOA of central charge c such that its minimal non-zero conformal weight is greater than $[\frac{c}{24}]$ (see also [Hö08]). The extremal VOAs (CFTs) are studied in [Wit07] in relation to three-dimensional gravity. Inspired by [Wit07], the $\mathcal{N}=2$ and $\mathcal{N}=4$ extremal extremal elliptic genera are defined in [GGK+08].

[12] More precisely, the Ramond vacuum representation means the Ramond massless representation of conformal weight $\frac{c}{24}$ and $\mathrm{U}(1)$ charge $\frac{c}{6}$ [CDD+14, §7] (which is also called the Ramond graviton representation of the same weight and charge [Egu04]). The quantity (3.39) is obtained as the $\frac{1}{2}$-spectral flow of the character of the NS vacuum representation (the NS graviton representation of conformal weight 0 and $\mathrm{U}(1)$ charge 0 [Egu04]). This (3.39) is the terms of the character of the Ramond vacuum representation, containing all its polar terms.

are [GGK+08, BDFK15]

$$Z_{\rm ext}^{m=1}(\tau,z) = \varphi_{0,1}, \tag{3.40}$$
$$Z_{\rm ext}^{m=2}(\tau,z) = \frac{1}{6}\varphi_{0,1}^2 + \frac{5}{6}E_4\varphi_{-2,1}^2, \tag{3.41}$$
$$Z_{\rm ext}^{m=3}(\tau,z) = \frac{1}{48}\varphi_{0,1}^3 + \frac{7}{16}E_4\varphi_{-2,1}^2\varphi_{0,1} + \frac{13}{24}E_6\varphi_{-2,1}^3, \tag{3.42}$$
$$Z_{\rm ext}^{m=4}(\tau,z) = \frac{1}{432}\varphi_{0,1}^4 + \frac{1}{8}E_4\varphi_{-2,1}^2\varphi_{0,1}^2 + \frac{11}{27}E_6\varphi_{-2,1}^3\varphi_{0,1} + \frac{67}{144}E_4^2\varphi_{-2,1}^4. \tag{3.43}$$

3.4 Elliptic Genus

Weak Jacobi forms appear in physics as the elliptic genus of $\mathcal{N}=2$ SCFT. (Be careful not to confuse it with the extremal $\mathcal{N}=2$ elliptic genus, which was purely mathematically defined as above.) To explain what it is, let us begin with the Witten index of a chiral $\mathcal{N}=1$ SCFT.

In the R sector of a chiral $\mathcal{N}=1$ SCFT of central charge c, there is a supercharge G_0 such that $(G_0)^2 = L_0 - \frac{c}{24}$ (see (4.23) below). We then have, for a general state $|h\rangle$ with L_0-eigenvalue h in the R sector,

$$|G_0|h\rangle|^2 = \langle h|(L_0 - \frac{c}{24})|h\rangle = h - \frac{c}{24}, \tag{3.44}$$

where G_0 is a Hermitian operator. As a result, unless $h = \frac{c}{24}$, $G_0|h\rangle$ is a non-zero state. Since $[G_0, L_0] = 0$ and $\{G_0, (-1)^F\} = 0$, where $(-1)^F$ is the fermion parity operator, we can say that $G_0|h\rangle$ is a state with the same L_0-eigenvalue and opposite parity compared with $|h\rangle$, when $h \neq \frac{c}{24}$. Therefore, $|h\rangle$ and $G_0|h\rangle$ cancel in the trace over the R sector $\mathcal{H}_{\rm R}$

$$\mathrm{Tr}_{\mathcal{H}_{\rm R}}(-1)^F q^{L_0-\frac{c}{24}}, \tag{3.45}$$

and only the states with $h = \frac{c}{24}$ contribute. So the quantity (3.45) is a constant. A state with $h = \frac{c}{24}$ in the R sector is called a *Ramond vacuum*,[13] and the constant (3.45) is counting the number of Ramond vacua with sign $(-1)^F$. This constant is called the *Witten index*. (We can also define the Witten index for a general (not necessarily conformal) supersymmetric theory in a similar way.)

The elliptic genus is a concept similar to the Witten index. We define the *elliptic genus* of an $\mathcal{N}=(0,1)$ SCFT which is non-chiral (consisting of left- and right-moving parts) as

$$Z_{\rm ell}(\tau) := \mathrm{Tr}_{\mathcal{H}_{\rm R\tilde{R}}}(-1)^{F+\tilde{F}} q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}, \tag{3.46}$$

where $\mathcal{H}_{\rm R\tilde{R}}$ is the Hilbert space of the entire R sector, and the objects without tilde and with tilde $\tilde{}$ denote the objects in left- and right-moving parts respectively. The entire R sector has the structure $\mathcal{H}_{\rm R\tilde{R}} = \bigoplus_i \mathcal{H}_{{\rm R},i} \otimes \mathcal{H}_{\tilde{\rm R},i}$, and by the discussion similar to above, we can say that the

[13] Be careful not to confuse $|0\rangle_{\rm R}$ introduced in (6.20) with a Ramond vacuum. In fact, the conformal weight of $|0\rangle_{\rm R}$ of the n real free chiral fermions is $\frac{n}{16} = \frac{c}{8}$, and the theory is not supersymmetric.

sectors containing the right-moving Ramond vacua $\bigoplus_{i \text{ such that } \mathcal{H}_{\widetilde{\mathrm{R}},i}=\mathrm{vac}} \mathcal{H}_{\mathrm{R},i} \otimes \mathcal{H}_{\widetilde{\mathrm{R}},i}$ only contribute to the trace (3.46). Therefore, the elliptic genus $Z_{\mathrm{ell}}(\tau)$ is independent of $\bar{q}$.

We regard a non-chiral $\mathcal{N} = (\mathcal{N}_l, \mathcal{N}_r)$ SCFT with $\mathcal{N}_l \geq 0$ and $\mathcal{N}_r \geq 1$ as a special case of non-chiral $\mathcal{N} = (0,1)$ SCFTs. If $\mathcal{N}_l \geq 1$, then it further follows that the elliptic genus $Z_{\mathrm{ell}}(\tau)$ is independent of q, and hence a constant. We may then call it the Witten index.

If $\mathcal{N}_l \geq 2$, then the quantity

$$Z_{\mathrm{ell}}(\tau, z) := \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}\widetilde{\mathrm{R}}}}(-1)^{F+\tilde{F}} y^{J_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\widetilde{L}_0 - \frac{\tilde{c}}{24}}, \tag{3.47}$$

where J_0 is the $\mathrm{U}(1)$ charge (the zero mode of $J(z)$) of the $\mathcal{N} = 2$ superconformal algebra,[14] is also called the *elliptic genus*. In fact, this (3.47) is the traditional definition of the elliptic genus, but recently, the scope of the term "elliptic genus" seems to slightly broadened so that it includes the quantity $Z_{\mathrm{ell}}(\tau) = Z_{\mathrm{ell}}(\tau, 0)$ in (3.46).

Moreover, for a fermionic chiral CFT, by regarding it as coupling to a trivial right-moving $\mathcal{N}_r = 1$ theory, its elliptic genus is

$$Z_{\mathrm{ell}}(\tau) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}}}(-1)^F q^{L_0 - \frac{c}{24}}. \tag{3.55}$$

If the chiral CFT has $\mathcal{N} = 1$ supersymmetry, then this is just the Witten index. If $\mathcal{N} \geq 2$, then the quantity

$$Z_{\mathrm{ell}}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}}}(-1)^F y^{J_0} q^{L_0 - \frac{c}{24}}, \tag{3.56}$$

is also the elliptic genus.

Let us come back to the weak Jacobi forms. We consider a non-chiral $\mathcal{N} = (2,1)$ SCFT of central charge $(c, \tilde{c})$ or a chiral $\mathcal{N} = 2$ SCFT of central charge c, such that the phases caused by

[14]The $\mathcal{N} = 2$ *superconformal algebra* of central charge c consists of the operators $T(z), J(z), G^+(z), G^-(z)$ satisfying the following OPEs:

$$T(z_1)T(z_2) \sim \frac{c/2}{(z_1 - z_2)^4} + \frac{2}{(z_1 - z_2)^2} T(z_2) + \frac{1}{z_1 - z_2} \partial T(z_2), \tag{3.48}$$

$$T(z_1)J(z_2) \sim \frac{1}{(z_1 - z_2)^2} J(z_2) + \frac{1}{z_1 - z_2} \partial J(z_2), \tag{3.49}$$

$$T(z_1)G^\pm(z_2) \sim \frac{3/2}{(z_1 - z_2)^2} G^\pm(z_2) + \frac{1}{z_1 - z_2} \partial G^\pm(z_2), \tag{3.50}$$

$$J(z_1)J(z_2) \sim \frac{c/3}{(z_1 - z_2)^2}, \tag{3.51}$$

$$J(z_1)G^\pm(z_2) \sim \pm \frac{1}{z_1 - z_2} G^\pm(z_2), \tag{3.52}$$

$$G^+(z_1)G^-(z_2) \sim \frac{2c/3}{(z_1 - z_2)^3} + \frac{2}{(z_1 - z_2)^2} J(z_2) + \frac{1}{z_1 - z_2}(2T(z_2) + \partial J(z_2)), \tag{3.53}$$

$$G^\pm(z_1)G^\pm(z_2) \sim 0. \tag{3.54}$$

These operators, $T(z)$, $J(z)$, and $G^\pm(z)$ are called the *energy-momentum tensor*, the $\mathrm{U}(1)$ *current*, and the *supercurrents*, respectively.

the gravitational anomaly in the modular transformations (E.1, E.2) are trivial, that is, $2(\tilde{c} - c) \equiv 0 \bmod 24$. If the $\mathrm{U}(1)$ charges (the J_0-eigenvalues) of the states in the NS sector of the SCFT are all integers, and $\frac{c}{3}$ is a positive integer, then the elliptic genus $Z_{\mathrm{ell}}(\tau, z)$ in (3.47) or (3.56) is known [KYY93] to be a weak Jacobi form of weight 0 and index $\frac{c}{6}$. (There is also the concept of a weak Jacobi form of half-integer index.) The transformation (3.26) follows from the modular invariance, and the transformation (3.27) follows from the spectral flow.

We have defined the extremal $\mathcal{N} = 2$ elliptic genera in Section 3.3. Constructing an SCFT whose elliptic genus coincides with an extremal $\mathcal{N} = 2$ elliptic genus is a nontrivial problem. For example, $Z_{\mathrm{ext}}^{m=1}(\tau, z)$ is the elliptic genus of the K3 CFT (Section 7) divided by 2. Duncan's module will be introduced as an $\mathcal{N} = 1$ chiral SCFT in Section 6.2, but it also admits $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal algebras, and its elliptic genus as an $\mathcal{N} = 2$ theory is $Z_{\mathrm{ext}}^{m=2}(\tau, z)$ [CDD+14]. An $\mathcal{N} = 2$ chiral SCFT with elliptic genus $Z_{\mathrm{ext}}^{m=4}(\tau, z)$ is constructed in [BDFK15] from the odd Leech lattice. There is also the concept of $\mathcal{N} = 4$ extremal elliptic genera $Z_{\mathrm{ext}}^{m,\mathcal{N}=4}(\tau, z)$ [GGK+08]. Duncan's module as $\mathcal{N} = 4$ theory has the elliptic genus $Z_{\mathrm{ext}}^{m=2,\mathcal{N}=4}(\tau, z)$. An $\mathcal{N} = 4$ chiral SCFT with $Z_{\mathrm{ext}}^{m=4,\mathcal{N}=4}(\tau, z)$ is constructed in [Har16]. See [FH17, KY23b] for more on SCFTs with extremal elliptic genera.

4 Vertex Operator Algebras

The axiomatic definition of a VOA is somewhat technical. In many cases for physicists, it suffices to consider the examples of VOAs introduced in the next Part II, whose constructions would be more familiar from the viewpoint of CFTs in physics. We will not go into the proof that these examples actually satisfy the axioms of a VOA. Even so, reviewing the definition of a VOA is more or less helpful for understanding how VOAs mathematically formulate CFTs in physics, so we will summarize it in this Section 4.

The historical development of the concept of VOA is summarized in [FLM88, Introduction §III]. It developed through three stages: vertex operator, vertex algebra, and vertex operator algebra.

Vertex operator (a historical note)

Lepowsky and Wilson constructed a certain representation of the affine Lie algebra $\widehat{\mathfrak{sl}}(2,\mathbb{C})$ in [LW78], using the operators which are now called *twisted vertex operators*, which was generalized to other affine Lie algebras in [KKLW81]. Garland noticed the similarity between those operators and the vertex operators in a physics theory called the dual resonance theory, and it was confirmed that this resemblance can actually be made into a complete coincidence in the works by Frenkel and Kac [FK80], and by Segal [Seg81] independently, where representations of the affine Lie algebras were constructed using the (untwisted) *vertex operators*.

In [FLM84], Frenkel, Lepowsky, and Meurman constructed a representation, which we now call the moonshine module $V^\natural$, of a certain algebra $\hat{B}^\natural$, an "affinization" of the Griess algebra $B^\natural$, using vertex operators. They also showed that the monster group $\mathbb{M}$ acts on $V^\natural$, and the graded character of $V^\natural$ is the modular j-function (without the constant term).

4.1 Vertex Algebras

Motivated by [FLM84], Borcherds [Bor86] introduced the axioms of a *vertex algebra*.[15] Following [Har99, §5.2], they can be summarized as follows.

Definition 4.1 (vertex algebra)**.** A vector space V over a field $\mathbb{K}$ of characteristic 0 is called a *vertex algebra* if it satisfies the following axioms.

(1) For any $v \in V$ and any $n \in \mathbb{Z}$, a $\mathbb{K}$-linear endomorphism $v_{(n)} \in \mathrm{End}(V)$ is given.

We define a formal series $Y(v,z) \in \mathrm{End}(V)[[z,z^{-1}]]$ of z, called the *vertex operator* corresponding to v, as

$$Y(v,z) := \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}. \tag{4.1}$$

[15] Borcherds posted on MathOverflow that his definition of a vertex algebra was purely motivated by an attempt to understand the works by Frenkel, Lepowsky, and Meurman, and he did not use any insights from field theories in physics, simply because he was barely familiar with such topics. https://mathoverflow.net/questions/53988/what-is-the-motivation-for-a-vertex-algebra

The map $V \to \mathrm{End}(V)[[z, z^{-1}]]; v \mapsto Y(v, z)$ is often called the *state-field correspondence*, and we require it to be $\mathbb{K}$-linear.

(2) For any $v, w \in V$, there exists $n \in \mathbb{Z}$ such that $v_{(m)}w = 0$ for any $m \geq n$.

(3) For any $u, v, w \in V$ and $m, n, q \in \mathbb{Z}$,

$$\sum_{i\geq 0} \binom{m}{i} (u_{(q+i)}v)_{(m+n-i)}w = \sum_{i\geq 0} (-1)^i \binom{q}{i} [u_{(m+q-i)}(v_{(n+i)}w) - (-1)^q v_{(n+q-i)}(u_{(m+i)}w)]. \tag{4.2}$$

This equation (4.2) is called the *Borcherds identity* (or the *Jacobi identity*).

(4) There is a specific element $\mathbf{1} \in V$ called the *vacuum vector*, which satisfies for any $v \in V$, $v_{(-1)}\mathbf{1} = v$, and $v_{(n)}\mathbf{1} = 0$ for any $n \geq 0$.

■

Following physics convention, any element $v \in V$ of a vertex algebra V is called a *state*, and a vertex operator $Y(v, z)$ a *field*, a *current*, or simply an *operator*. Sometimes v is also called a current. The operator $v_{(n)}$ is called the *n-th mode* of the mode expansion of $Y(v, z)$, or simply of v.

The Borcherds identity (4.2) is equivalent to

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2)w - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1)w = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2)w, \tag{4.3}$$

where

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n, \tag{4.4}$$

and we break down

$$\delta\left(\frac{z_1 - z_2}{z_0}\right) = \sum_{n\in\mathbb{Z}} \left(\frac{z_1 - z_2}{z_0}\right)^n \tag{4.5}$$

into an infinite sum of the terms in the form of $z_0^i z_1^j z_2^k$, under the definition

$$(a - b)^n := a^n(1 - \frac{b}{a})^n := \sum_{i=0}^{\infty} \binom{n}{i} a^{n-i}(-b)^i \quad \text{for } n < 0. \tag{4.6}$$

In particular, $(a - b)^n$ is not equal to $(-b + a)^n$ if $n < 0$, under this definition.

There are some variants of the definitions of vertex algebras. For example, [Kac98, CKLW15] adopt the following definition. We will use[16] the notation $[A, B] := AB - BA$.

[16] [Kac98] deal with vertex superalgebras, rather than vertex algebras, throughout the entire book. The definition of the bracket there is $[A, B] := AB - (-1)^{|A||B|}BA$. See also the description of vertex superalgebras in the main text below.

Definition 4.2 (vertex algebra (another definition))**.** A vector space V over $\mathbb{C}$ is called a *vertex algebra* if it satisfies the following axioms.

(1) For any $v \in V$ and any $n \in \mathbb{Z}$, a $\mathbb{C}$-linear endomorphism $v_{(n)} \in \mathrm{End}(V)$ is given.

The formal series $Y(v,z) := \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$ is called the *vertex operator* corresponding to v, and the *state-field correspondence* $V \to \mathrm{End}(V)[[z,z^{-1}]]; v \mapsto Y(v,z)$ is required to be a linear map.

(2) For any $u, v \in V$, there exists $n \in \mathbb{Z}$ such that $u_{(m)}v = 0$ for any $m \geq n$.

A formal series $\sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in \mathrm{End}(V)[[z,z^{-1}]]$ such that for any $w \in V$, $a_n w = 0$ for sufficient large k is called a *field*. The vertex operators $Y(v,z)$ are fields, and the state-field correspondence is a linear map from V to the space of fields.

(3) For any $v, w \in V$, there exists $N \in \mathbb{Z}_{>0}$ such that

$$(z_0 - z_1)^N [Y(v,z_0), Y(w,z_1)] = 0. \tag{4.7}$$

This condition is called the *locality*.

(4) There is a specific operator $T \in \mathrm{End}(V)$ called the *infinitesimal translation operator*, which satisfies

$$[T, Y(v,z)] = \frac{d}{dz} Y(v,z) \tag{4.8}$$

for any $v \in V$. This condition is called the *translation covariance*.

(5) There is a specific element $\mathbf{1} \in V$ called the *vacuum vector*, which satisfies $T\mathbf{1} = 0$, $Y(\mathbf{1},z) = \mathrm{id}_V$, and $v_{(-1)}\mathbf{1} = v$ for any $v \in V$.

■

Even if we replace the condition $v_{(-1)}\mathbf{1} = v$ with

$$Y(v,z)\mathbf{1}|_{z=0} = v, \tag{4.9}$$

we obtain an equivalent definition [CKLW15, Remark 4.1]. We can also show [Kac98, Cor. 4.4 (c)]

$$[T, Y(v,z)] = Y(Tv,z) = \frac{d}{dz} Y(v,z). \tag{4.10}$$

A vertex algebra of this Definition 4.2 satisfies the Borcherds identity (4.2) of Definition 4.1 [Kac98, §4.8].

4.2 Vertex Operator Algebras

Finally, Frenkel, Lepowsky, and Meurman [FLM88] introduced the axioms of a *vertex operator algebra*. Following [Har99, §5.2], they can be summarized as follows.

Definition 4.3 (vertex operator algebra)**.** A vector space V over a field $\mathbb{K}$ of characteristic 0 is called a *vertex operator algebra (VOA)* if it satisfies the following axioms.

(1) V is a vertex algebra in the sense of Definition 4.1, with the state-field correspondence $v \mapsto Y(v, z)$ and the vacuum vector $\mathbf{1}$.

(2) $Y(\mathbf{1}, z) = \mathrm{id}_V$.

(3) There is a specific element $\omega \in V$ called the *Virasoro element* such that if we define $L_n := \omega_{(n+1)}$, that is,

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{4.11}$$

then $\{L_n\}_{n\in\mathbb{Z}}$ satisfy the commutation relations of the *Virasoro algebra*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \tag{4.12}$$

where $c \in \mathbb{K}$ is called the *central charge*.

(4) L_0 is diagonalizable on V, and its eigenvalues lie in $\mathbb{Z}$. So the eigenspace decomposition of V with respect to L_0 defines a $\mathbb{Z}$-grading on V as $V = \bigoplus_{h\in\mathbb{Z}} V_h$.

If $v \in V_h$, then v is called a *homogeneous* element of *(conformal) weight* or *(conformal) dimension* $h =: \mathrm{wt}(v)$.

(5) Every subspace V_h is finite-dimensional, and there is $h_{\min} \in \mathbb{Z}$ such that $V_h = 0$ for any $h < h_{\min}$.

(6) For any $v \in V$,

$$Y(L_{-1}v, z) = \frac{d}{dz} Y(v, z). \tag{4.13}$$

■

In addition, a VOA is said to be *of CFT type* [DLMM98], if $h_{\min} = 0$ and $V_0 = \mathbb{K}\mathbf{1}$.

Following physics convention, the operator $T(z) := Y(\omega, z)$ is called the *energy-momentum tensor*. We can show $\mathrm{wt}(\omega) = 2$, because $L_0\omega = L_0L_{-2}\mathbf{1} = 2L_{-2}\mathbf{1} = 2\omega$, using the Definition 4.1 of the vacuum vector $\mathbf{1}$ and (4.12).

There are some variants of the definitions of VOAs. For example, the corresponding concept in [Kac98, CKLW15] is a *conformal vertex algebra*, which is defined as a vertex algebra V in the sense of Definition 4.2, with a specific element $\omega \in V$ such that $\{L_n\}_{n\in\mathbb{Z}}$ defined as in (4.11) satisfy (4.12), L_0 is diagonalizable on V (which defines a $\mathbb{C}$-grading on V in general), and $L_{-1} = T$ (compare (4.10) and (4.13)).

Here is one caveat on the convention of the weight. From the axioms of VOAs, we can show [Kac98, Eq. (4.9.3) and Theorem 4.10 (e)] that for a homogeneous $v \in V$,

$$[L_0, v_{(n)}] = -(n+1-\mathrm{wt}(v))v_{(n)}. \tag{4.14}$$

Therefore, $v_{(n)}$ maps V_h to $V_{h-(n+1-\mathrm{wt}(v))}$. In physics literature, however, the usual convention is

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n;\mathrm{phys})} z^{-n-\mathrm{wt}(v)}, \tag{4.15}$$

and hence

$$v_{(n)} = v_{(n+1-\mathrm{wt}(v);\mathrm{phys})}. \tag{4.16}$$

Some mathematics literature e.g. [Kac98, CKLW15] also introduces this convention. Then

$$[L_0, v_{(n;\mathrm{phys})}] = -n v_{(n;\mathrm{phys})}, \tag{4.17}$$

and therefore $v_{(n;\mathrm{phys})}$ maps V_h to V_{h-n}.

More on vertex algebras and vertex operator algebras

If a subspace W of a vertex algebra or a VOA V forms a vertex algebra or a VOA under the same structure as V, then W is called a *vertex subalgebra* or a *sub-VOA* of V, respectively.

A *module* or a *representation* of a vertex algebra V is a vector space M such that for any $v \in V$, a field

$$Y^M(v,z) = \sum_{n\in\mathbb{Z}} v^M_{(n)} z^{-n-1}, \quad v^M_{(n)} \in \mathrm{End}(M), \tag{4.18}$$

is given, the state-field correspondence $v \mapsto Y^M(v,z)$ is linear, $Y^M(\mathbf{1},z) = \mathrm{id}_M$, and the Borcherds identity (4.2) for them holds. A vertex algebra V itself is a module of V, and sometimes called the *adjoint module*. A *($\mathbb{Z}_N$-)twisted module* of a vertex operator uses $\mathrm{End}(M)[[z^{\frac{1}{N}}, z^{-\frac{1}{N}}]]$ ($N \in \mathbb{Z}_{>0}$) instead of $\mathrm{End}(M)[[z,z^{-1}]]$. Some literature [DL93] also introduces a concept we may call "twisted vertex algebras", which use $\mathrm{End}(V)[[z^{\frac{1}{N}}, z^{-\frac{1}{N}}]]$ instead of $\mathrm{End}(V)[[z,z^{-1}]]$.

A *vertex superalgebra* introduces an additional $\mathbb{Z}_2$-grading $V = V^{\bar{0}} \oplus V^{\bar{1}}$ called the *parity* to a vertex algebra, and the Borcherds identity (4.3) is generalized to the super version

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)w - (-1)^{|u||v|}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)w = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2)w, \tag{4.19}$$

where $|v| := 0, 1$ for $v \in V^{\bar{0}}, V^{\bar{1}}$, respectively. Correspondingly, the locality (4.7) is generalized to

$$(z_0 - z_1)^N (Y(v, z_0)Y(w, z_1) - (-1)^{|v||w|} Y(w, z_1)Y(v, z_0)) = 0. \tag{4.20}$$

[DL93] further introduces *generalized vertex algebras*, which subsume both twisted vertex algebras and vertex superalgebras as special cases.

A *vertex operator superalgebra (VOSA)* is a VOA with the parity $\mathbb{Z}_2$-grading $V = V^{\bar{0}} \oplus V^{\bar{1}}$ making it a vertex superalgebra, and the weight $\frac{1}{2}\mathbb{Z}$-grading $V = \bigoplus_{h \in \frac{1}{2}\mathbb{Z}} V_h$ instead of the $\mathbb{Z}$-grading. Sometimes $V^{\bar{0}} = \bigoplus_{h \in \mathbb{Z}} V_h$ and $V^{\bar{1}} = \bigoplus_{h \in \mathbb{Z}+\frac{1}{2}} V_h$ are further required [DMC14, §2.1]. Despite of its name "super," a VOSA just describes a fermionic CFT in physics, not necessarily with a supersymmetry. For the description of SCFT in physics, we introduce the following structures.

An $\mathcal{N} = 1$ *vertex operator superalgebra* requires the existence of a specific element $\tau \in V^{\bar{1}}_{\frac{3}{2}}$ such that if we define $G_r := \tau_{(r+\frac{1}{2})}$, that is,

$$Y(\tau, z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}}, \tag{4.21}$$

then $\{G_r\}_{r \in \mathbb{Z}+\frac{1}{2}}$ satisfy the $\mathcal{N} = 1$ superconformal algebra

$$[L_m, G_r] = \Big(\frac{1}{2}m - r\Big) G_{m+r}, \tag{4.22}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}\Big(r^2 - \frac{1}{4}\Big)\delta_{r,-s}, \tag{4.23}$$

together with (4.12). This element τ or its operator $G(z) := Y(\tau, z)$ is called the *supercurrent*. $\mathcal{N} = 2$ and $\mathcal{N} = 4$ *vertex operator superalgebras* are also defined in a similar way, by requiring the existence of elements whose corresponding fields satisfy the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal algebras, respectively.

As for more details on these definitions, see for example [DL93, Kac98, FBZ04] and references in [DMC14, §2.1].

VOAs formulate CFTs from the viewpoint of state-field correspondence, while there is another mathematical formulation of CFTs, called *local conformal nets*, which focuses on algebras of local operators. These two formulations are useful in different situations, and the translation between them is one of the research topics in the field of operator algebras. See for example [CKLW15, Kaw15, Kaw17].

Part II

Review of Moonshine Phenomena

This Part II is a review of moonshine phenomena and an introduction to important examples of VOAs. The most classical example of moonshine phenomena is the monstrous moonshine, and its underlying object, the monster VOA, is constructed as the $\mathbb{Z}_2$-orbifold of the Leech lattice VOA. So, in Section 5, after the review of the monstrous moonshine, we will introduce the general construction of lattice VOAs. Another well-established moonshine phenomenon is the Conway moonshine. In Section 6, we will review that the Conway moonshine module, also known as Duncan's module, is constructed from a Clifford module VOSA, which describes the CFT of free fermions. In the last Section 7, we will review a relatively new and notable example of moonshine phenomena, the K3 Mathieu moonshine, whose mysterious nature is not fully understood yet.

5 Monstrous Moonshine and Lattice VOA

The monstrous moonshine is the first discovered example of a moonshine phenomenon, and the origin of this research field. As we will review in Section 5.1, it was observed as an empirical relationship between the monster group $\mathbb{M}$ and the modular j-function $j(\tau)$, and theoretically explained by the existence of an underlying VOA called the monster VOA $V^\natural$. The monster VOA $V^\natural$ is the $\mathbb{Z}_2$-orbifold of the Leech lattice VOA. So we will review the general construction of lattice VOAs in Section 5.2, and their $\mathbb{Z}_2$-orbifolds in Section 5.3. The structures of the automorphism groups of these theories are also described.

5.1 Monstrous Moonshine

In 1978 and 1979, McKay and Thompson [Tho79b, McK01] observed that the first several coefficients (except for the constant term) of the modular j-function

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots \tag{5.1}$$

$$= \sum_{i=-1}^{\infty} c_i q^i, \tag{5.2}$$

can be written as simple sums of irreducible representation dimensions of the monster group $\mathbb{M}$ as follows. The irreducible representation dimensions of $\mathbb{M}$ are, from the smallest one,

i	1	2	3	4	5	6	7	$\cdots$
$\chi_i(1_{\mathbb{M}})$	1	196883	21296876	842609326	18538750076	19360062527	293553734298	$\cdots$

(5.3)

where χ_i denotes the i-th irreducible character of $\mathbb{M}$. The coefficients of the modular j-function are then

$$\begin{aligned}
c_1 &= 196884 = \chi_1(1) + \chi_2(1), && (5.4)\\
c_2 &= 21493760 = \chi_1(1) + \chi_2(1) + \chi_3(1), && (5.5)\\
c_3 &= 864299970 = 2\chi_1(1) + 2\chi_2(1) + \chi_3(1) + \chi_4(1), && (5.6)\\
c_4 &= 20245856256 = 3\chi_1(1) + 3\chi_2(1) + \chi_3(1) + 2\chi_4(1) + \chi_5(1) && (5.7)\\
&= 2\chi_1(1) + 3\chi_2(1) + 2\chi_3(1) + \chi_4(1) + \chi_6(1), && (5.8)\\
c_5 &= 333202640600 = 4\chi_1(1) + 5\chi_2(1) + 3\chi_3(1) + 2\chi_4(1) + \chi_5(1) + \chi_6(1) + \chi_7(1), && (5.9)\\
&\vdots
\end{aligned}$$

The constant term of the modular j-function is not important in the sense that, even if we change it, the key property of the modular j-function that it is the generator of the rational function field $\mathbb{C}(j)$ of the modular functions of weight 0 (see Thm. 3.2) does not change. So they conjectured that there exists a graded vector space $V = \bigoplus_{i=-1}^{\infty} V_i$ such that it is a representation of the monster group $\mathbb{M}$, and its graded character of the identity element $1_{\mathbb{M}}$ is the modular j-function without the constant term

$$J(\tau) := j(\tau) - 744. \tag{5.10}$$

In addition, this graded vector space V must have some nice property; otherwise, we can easily construct it by just stacking the trivial representation χ_1. So they focused on the graded characters

$$J_g(\tau) := \sum_{i=-1}^{\infty} \mathrm{Tr}_{V_i}(g) q^i \tag{5.11}$$

of the elements $g \in \mathbb{M}$ other than the identity $1_{\mathbb{M}}$. This character $J_g(\tau)$ is called the *McKay–Thompson series* associated with $g \in \mathbb{M}$.

Recall (Section 3.2) that $J_{1_{\mathbb{M}}}(\tau) = J(\tau)$ is the generator of the rational function field $\mathbb{C}(j)$ of the modular functions of weight 0. Such a modular function $f : \mathbb{H} \to \mathbb{C}$ is invariant under the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$, so it is well-defined as a function $f : \mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}$ on the quotient space $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. It is known that the one-point compactification (the compact space made by adding the point at infinity) of $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$, denoted by $(\mathbb{H}/\mathrm{SL}(2,\mathbb{Z}))^*$, is topologically equivalent to a Riemann sphere, which is a surface of genus 0. Pushing forward this observation (and an observation[17] by Ogg [Ogg75]), Thompson proposed the following conjecture in [Tho79a]: for each element $g \in \mathbb{M}$, there exists a subgroup Γ_g of the modular group $\mathrm{SL}(2,\mathbb{Z})$ (more precisely, a congruence subgroup) such that $(\mathbb{H}/\Gamma_g)^*$ is a genus-zero surface, and

[17] In 1975, Ogg already observed that a prime number p divides the order $|\mathbb{M}|$ of the monster group if and only if $(\mathbb{H}/\Gamma_0(p)^+)^*$ has genus 0, where $\Gamma_0(p)^+ = \langle \Gamma_0(p), \frac{1}{\sqrt{p}}\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \rangle$. This is an unofficial beginning of the monstrous moonshine [Gan06a].

$J_g(\tau)$ is the generator of the rational function field of the functions $(\mathbb{H}/\Gamma_g)^* \to \mathbb{C}$ over $\mathbb{C}$. This is the *McKay–Thompson conjecture*.

The McKay–Thompson conjecture was proven through the following progression. In [CN79], Conway and Norton proposed the conjectural forms of Γ_g and $J_g(\tau)$ for each element $g \in \mathbb{M}$. This is called the *Conway–Norton conjecture*, or the *monstrous moonshine conjecture*. In addition, they also showed that these conjectural Γ_g and $J_g(\tau)$ satisfy that $(\mathbb{H}/\Gamma_g)^*$ has genus 0 and $J_g(\tau)$ generates the rational function field of the functions $(\mathbb{H}/\Gamma_g)^* \to \mathbb{C}$. Frenkel, Lepowsky, and Meurman constructed a representation $V^\natural$ of the monster group $\mathbb{M}$ whose graded character is $J(\tau)$ in [FLM84], and clarified its VOA structure and showed that its automorphism group $\mathrm{Aut}(V^\natural)$ is precisely the monster group $\mathbb{M}$ in [FLM88]. Finally, Borcherds showed in [Bor92] that the graded character of $g \in \mathbb{M}$ on $V^\natural$ coincides with the conjectured $J_g(\tau)$ by the Conway–Norton conjecture, for which he was awarded the Fields Medal.

The $\mathbb{M}$-module $V^\natural$ constructed in [FLM88] is called the monster VOA or the moonshine module. The monster VOA $V^\natural$ is constructed as the $\mathbb{Z}_2$-orbifold of the Leech lattice VOA $V_{\Lambda_{24}}$, the details of which are reviewed in the following Sections 5.2 and 5.3. In the language of physics, $V^\natural$ is a chiral bosonic CFT of central charge 24, having the partition function $J(\tau)$ and the symmetry $\mathbb{M}$. Let us briefly review why this is the case.

In general, we can construct a chiral bosonic modular-invariant lattice CFT V_L from a Euclidean even self-dual lattice L of rank $n \equiv 0 \bmod 24$. Such a lattice CFT V_L has central charge n, and its partition function is

$$Z^{(V_L)}(\tau) := \mathrm{Tr}_{V_L}[q^{L_0 - \frac{c}{24}}] \tag{5.12}$$

$$= \frac{\Theta_L(\tau)}{\eta(\tau)^n}. \tag{5.13}$$

For this partition function $Z^{(V_L)}(\tau)$ to start with the term q^{-1}, we should choose $n = 24$. So the lattice must be an even self-dual lattice of rank 24, that is, a Niemeier lattice.

Now that $Z^{(V_L)}(\tau)$ is modular invariant (a modular function of weight 0) starting with the term q^{-1}, it must be in the form of $J(\tau)$ + constant, and in fact, we have

$$Z^{(V_L)}(\tau) = J(\tau) + 24 + |L_1|. \tag{5.14}$$

Here, $|L_1|$ is the number of lattice vectors with squared length 2 (as defined in (2.41)), and hence $|L_1| = 0$ for the Leech lattice $L = \Lambda_{24}$. In addition, 24 is coming from the states $\alpha^i_{-1}|0\rangle$ ($i = 1, \ldots, 24$) of conformal weight (L_0-eigenvalue) 1, and these states have odd parity under the $\mathbb{Z}_2$ symmetry $k \mapsto -k$ of the lattice L. Therefore, these 24 states are eliminated by taking the $\mathbb{Z}_2$-orbifold, which retains the $\mathbb{Z}_2$-invariant states. As a result, the partition function of the $\mathbb{Z}_2$-orbifold of the Leech lattice CFT $V_{\Lambda_{24}}$ coincides with the modular j-function without the constant term $J(\tau)$.

The proof of $\mathrm{Aut}(V^\natural) = \mathbb{M}$ by [FLM88] can be outlined as follows. As mentioned in Section 2.4, the monster group $\mathbb{M}$ is generated by a subgroup $C(\Lambda_{24}) = 2^{1+24}.\mathrm{Co}_1$ and a specific order-2 element σ. Recall that the isometry group of the Leech lattice Λ_{24} is the Conway group Co_0. Before taking the $\mathbb{Z}_2$-orbifold, the automorphism group $\mathrm{Aut}(V_{\Lambda_{24}})$ of the Leech lattice VOA is a group extension $2^{24}.\mathrm{Co}_0$ of Co_0 (Section 5.2.3). When we construct the $\mathbb{Z}_2$-orbifold $V^\natural$, we add the twisted sector $(V_{\Lambda_{24}})_{\mathrm{tw}}$ as $V_{\Lambda_{24}} \oplus (V_{\Lambda_{24}})_{\mathrm{tw}}$, and project them onto the $\mathbb{Z}_2$-invariant states $(V_{\Lambda_{24}})^0 \oplus (V_{\Lambda_{24}})^0_{\mathrm{tw}}$, which is $V^\natural$ (Section 5.3.1). The addition of the twisted sector makes the group $2^{24}.\mathrm{Co}_0$ acting on $V_{\Lambda_{24}}$ enlarge to $2^{1+24}.\mathrm{Co}_0$ acting on $V_{\Lambda_{24}} \oplus (V_{\Lambda_{24}})_{\mathrm{tw}}$, and the projection makes $2^{1+24}.\mathrm{Co}_0$ into its $\mathbb{Z}_2$-quotient $C(\Lambda_{24}) = 2^{1+24}.\mathrm{Co}_1$ (Section 5.3.2).

In this way, the subgroup $C(\Lambda_{24})$ acts on the monster VOA $V^\natural$ naturally, but the construction of the order-2 element σ is quite nontrivial. While $C(\Lambda_{24})$ acts on the untwisted sector $(V_{\Lambda_{24}})^0$ and the twisted sector $(V_{\Lambda_{24}})^0_{\mathrm{tw}}$ separately, $\sigma \in \mathrm{Aut}(V^\natural)$ was constructed as an automorphism mixing these two sectors, and called the triality operator in [FLM88]. Its construction was revisited by [DGM90a, DGM94], and explained in relation to an isomorphism between two $\mathbb{Z}_2$-orbifolds by different kinds of $\mathbb{Z}_2$ symmetries (Section 5.3.3). Finally, by showing that $\mathbb{M}$ generated by $C(\Lambda_{24})$ and σ is the whole automorphism group $\mathrm{Aut}(V^\natural)$, we have $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$. We also note that another simple construction of $V^\natural$ and proof of $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$ were provided by Miyamoto in [Miy04].

Before proceeding, we remark that the automorphism group $\mathrm{Aut}(V)$ of a VOA V preserves the Virasoro algebra (4.12) of V by its definition, so we should be able to decompose the partition function $J(\tau)$ of $V^\natural$ simultaneously into the characters of the monster group $\mathbb{M}$ and the characters of the Virasoro algebra. The irreducible character[18] $\mathrm{ch}_h(\tau)$ of the Virasoro algebra of highest weight h and central charge $c = 24$ is

$$\mathrm{ch}_0(\tau) = \frac{q^{1/24}}{\eta(q)}(q^{-1} - 1), \tag{5.15}$$

$$\mathrm{ch}_h(\tau) = \frac{q^{1/24}}{\eta(q)} q^{h-1} \quad (h \in \mathbb{Z}_{>0}). \tag{5.16}$$

Here, we have

$$\frac{q^{1/24}}{\eta(q)} = \frac{1}{\prod_{m=1}^{\infty}(1-q^m)} = \sum_{N=0}^{\infty} p(N) q^N \tag{5.17}$$

$$= 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 \cdots , \tag{5.18}$$

[18] We can generate a module V_h of the Virasoro algebra from the highest weight state $|h\rangle$ (a state defined as $L_0|h\rangle = h|h\rangle$ and $L_m|h\rangle = 0$ for any $m > 0$) as $V_h := \mathrm{Span}_{\mathbb{C}}\{L_{-m_1} \cdots L_{-m_l}|h\rangle \mid m_1, \ldots, m_l \in \mathbb{Z}_{>0}\}$. Since the state $L_{-m_1} \cdots L_{-m_l}|h\rangle$ has L_0-eigenvalue $m_1 + \cdots + m_l + h$, the character of this module V_h is $\mathrm{Tr}_{V_h}[q^{L_0 - \frac{c}{24}}] = q^{h-\frac{c}{24}} \sum_{N=0}^{\infty} p(N) q^N = \frac{q^{1/24}}{\eta(q)} q^{h-\frac{c}{24}}$, using (5.17). However, this module V_h is not necessarily irreducible. When $h = 0$, the state $L_{-1}|0\rangle$ satisfies the defining properties of the highest weight state with $h = 1$ (for example, $L_1(L_{-1}|0\rangle) = (2L_0 - L_{-1}L_1)|0\rangle = 0$), so the module $V_{h=0}$ contains the submodule $V_{h=1}$. Therefore, the irreducible module of highest weight 0 has the character $\frac{q^{1/24}}{\eta(q)} q^{-\frac{c}{24}}(1-q)$. Whether the module V_h with $h > 0$ is irreducible or not depends on the values of weight h and central charge c, but for $c > 1$, it is known to be irreducible. See for example [DFMS97, Ch. 7], [ES15, Ch. 2] for more details.

where $p(N)$ denotes the number of the partitions of the integer N (the *partition function* in number theory). Therefore, $J(\tau)$ is decomposed as (note that $\mathrm{ch}_h(\tau)$ starts with the term q^{h-1})

$$J(\tau) = \sum_{i=-1}^{\infty} c'_i \mathrm{ch}_{i+1}(\tau), \tag{5.19}$$

where $c'_0 = 0$ and

$$\begin{aligned}
c'_{-1} &= \chi_1(1), && (5.20)\\
c'_1 &= \chi_2(1), && (5.21)\\
c'_2 &= \chi_3(1), && (5.22)\\
c'_3 &= \chi_4(1), && (5.23)\\
c'_4 &= \chi_6(1), && (5.24)\\
c'_5 &= \chi_5(1) + \chi_7(\tau), && (5.25)\\
&\vdots
\end{aligned}$$

We can see that the same representation χ_i of $\mathbb{M}$ appearing in the different coefficients c_i (5.4)–(5.9) are bundled into the representation of the Virasoro algebra, and the appearance of the irreducible representation dimensions of $\mathbb{M}$ becomes simple.

Lastly, we briefly mention a generalization of the monstrous moonshine. In the language of physics, the McKay–Thompson series (5.11) is an $\mathbb{M}$-twisted partition function of $V^\natural$ with a twisted boundary condition in the temporal direction. From the perspective of the general theory of orbifolds (Section F.2), it is also natural to consider the twisted partition functions with twisted boundary conditions in the spatial direction, or both spatial and temporal directions. (We remark that the temporal and spatial twists are referred to as *twining* and *twisted*, respectively, in the terminology of [GHV10a, GPRV12]. In physics literature, both of them are usually called *twisted*.) Moonshine conjectures related to these twisted partitions were proposed by Norton in [Nor87], by generalizing the work by Queen [Que81], and called the *generalized moonshine conjecture*. After many steps by mathematicians (see for example [Car17]), the final step of the proof of this generalized moonshine conjecture was announced in [Car12]. For more on monstrous moonshine and its recent developments, see for example [Har99, Gan06a, Gan06b, DGO14, HHP22] and references therein.

5.2 Lattice VOA

One typical construction of a VOA uses a lattice as an ingredient, and the resulting VOA is called a lattice VOA. Before getting into its mathematical description, we first present a brief explanation in the language of physics in Section 5.2.1. Then we see the construction of a lattice VOA in Section 5.2.2. We also describe the automorphism group of the lattice VOA in Section 5.2.3, because it plays an important role in moonshine phenomena.

5.2.1 Physical Description

For a given lattice L, we can consider a CFT having L as a momentum lattice. If the symmetric bilinear form of the lattice is of signature (r, s), and if it is diagonalized with respect to a certain basis $e_1, \dots, e_r, e_{r+1}, \dots, e_{r+s}$ as

$$|k|^2 = |k_+|_+^2 - |k_-|_-^2 \quad \text{for } k = k_+ + k_- = \sum_{i=1}^{r} k^i e_i + \sum_{i=r+1}^{r+s} k^i e_i \in L, \tag{5.26}$$

where $|\bullet|_+^2$ and $|\bullet|_-^2$ are some positive norms, then the partition function of the lattice CFT is

$$Z^{(V_L)}(\tau, \bar{\tau}) = \frac{1}{\eta(\tau)^r \overline{\eta(\tau)^s}} \Theta_L(\tau, \bar{\tau}) \tag{5.27}$$

$$= \frac{1}{\eta(\tau)^r \overline{\eta(\tau)^s}} \sum_{k \in L} q^{\frac{1}{2}|k_+|_+^2} \bar{q}^{\frac{1}{2}|k_-|_-^2}, \tag{5.28}$$

and the central charge of the lattice CFT is $(c, \tilde{c}) = (r, s)$. If the lattice L is even self-dual, then the partition function $Z^{(V_L)}(\tau, \bar{\tau})$ is bosonic and modular invariant[19] up to the phases (E.1, E.2) from the gravitational anomaly. An even self-dual lattice of signature (r, s) exists if and only if $r - s \equiv 0 \bmod 8$ [Ser73, Ch. V], so the phases from the gravitational anomaly vanish if the lattice L further satisfies $r - s \equiv 0 \bmod 24$, and then the partition function $Z^{(V_L)}(\tau, \bar{\tau})$ is completely modular invariant. If the lattice is Euclidean, then the resulting CFT is chiral. See for example [ES15, §5.5] for more details.

We will mainly deal with the chiral cases in these notes. The states of a chiral lattice CFT are $\mathbb{C}$-linear combinations of the states in the form of

$$\alpha_{-m_1}^{i_1} \cdots \alpha_{-m_l}^{i_l} |k\rangle, \tag{5.29}$$

where $|k\rangle$ is the state with momentum vector $k \in L$, $m_1, \dots, m_l \in \mathbb{Z}_{>0}$, and $\alpha_m^1, \dots, \alpha_m^n$ ($m \in \mathbb{Z}_{>0}$) are the creation operators corresponding to an $\mathbb{Z}$-basis $e_1, \dots, e_n$ of L. Together with the annihilation operators α_m^i ($m \in \mathbb{Z}_{<0}$) and the momentum operators α_0^i such that the state (5.29) is an eigenstate of α_0^i with eigenvalue $\langle e_i, k\rangle$, the operators $\{\alpha_m^i\}_{m\in\mathbb{Z}}^{i=1,\dots,n}$ satisfy the commutation relations $[\alpha_m^i, \alpha_{m'}^{i'}] = \langle e_i, e_{i'}\rangle m \delta_{m+m',0}$. The conformal weight of the state (5.29) is $\frac{1}{2}|k|^2 + m_1 + \cdots + m_l$. We also often use the creation-annihilation operators $\mathbb{a}_m^1, \dots, \mathbb{a}_m^n$ with respect to an orthonormal basis $\mathbb{e}_1, \dots, \mathbb{e}_n$ of $\mathbb{R}^n$ where the lattice L is embedded. They are related to $\alpha_m^1, \dots, \alpha_m^n$ under a proper $\mathbb{R}$-linear transformation, and their commutation relation is of course $[\mathbb{a}_m^i, \mathbb{a}_{m'}^{i'}] = \delta_{i,i'} m \delta_{m+m',0}$.

If we introduce the chiral bosons $X(z) = (X^i(z))_{i=1,\dots,n}$ which satisfy the OPE

$$X^i(z_1) \cdot X^j(z_2) \sim -\delta_{i,j} \log(z_1 - z_2), \tag{5.30}$$

[19] This follows from the formula (see e.g. [ES15, Eq. (5.166)]) of the modular S transformation $\Theta_L(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = \frac{1}{\det G}(-\sqrt{-1}\tau)^{r/2}(\sqrt{-1}\bar{\tau})^{s/2}\Theta_{L^*}(\tau, \bar{\tau})$, where G is the Gram matrix of L, and L^* is the dual lattice of L. The modular T transformation of $\Theta_L(\tau, \tau)$ is easy.

and whose mode expansions are given as (see e.g. [ES15, §1.8.1])

$$\partial X^i(z) = -\sqrt{-1} \sum_{m=-\infty}^{\infty} \frac{\alpha^i_m}{z^{m+1}}, \tag{5.31}$$

then we can describe the state-operator correspondence as (see e.g. [Pol07, §2.8])

$$\alpha^i_{-m}|0\rangle \quad \longleftrightarrow \quad \frac{\sqrt{-1}}{(m-1)!}\partial^m X^i(z) \quad (m \geq 1), \tag{5.32}$$

$$|k\rangle \quad \longleftrightarrow \quad V_k(z) \propto :e^{\sqrt{-1}k\cdot X(z)}:, \tag{5.33}$$

where $\propto$ denotes that we ignore the cocycle factor.

The details of the cocycle factors are described in Appendix C, but here we give a brief exposition. To realize the appropriate commutation relations of the vertex operators $V_k(z) \propto :e^{\sqrt{-1}k\cdot X(z)}:$, in accordance with whether $V_k(z)$ is bosonic (the weight $\frac{1}{2}|k|^2$ is an integer) or fermionic (a half-integer), we have to introduce a correction factor $c_k(p)$ satisfying

$$c_k(p+k')c_{k'}(p) = (-1)^{k\cdot k'+|k|^2|k'|^2}c_{k'}(p+k)c_k(p) = \varepsilon(k,k')c_{k+k'}(p) \tag{5.34}$$

to modify the commutation relations of $:e^{\sqrt{-1}k\cdot X(z)}:$'s. As a result, an additional factor called a *cocycle factor* $\varepsilon : L \times L \to \{\pm 1\}$ which satisfies the 2-cocycle condition appears in the OPE of the vertex operators $V_k(z) = :e^{\sqrt{-1}k\cdot X(z)}: c_k(p)$ as

$$V_k(z_1)\cdot V_{k'}(z_2) = (-1)^{|k|^2|k'|^2}V_{k'}(z_2)\cdot V_k(z_1) \sim \varepsilon(k,k')(z_1-z_2)^{k\cdot k'}V_{k+k'}(z_2), \tag{5.35}$$

where we dropped $O((z_1-z_2)^{k\cdot k'+1})$ terms in OPE.

5.2.2 Construction of Lattice VOA

The foundational literature on lattice VOAs is [FLM88], and [Kac98, §§5.4-5.5] also provides a detailed description. [DN99,Lam20] contain readable summaries, and we will follow them. These references except for [Kac98] deal with ordinary lattice VOAs constructed from even lattices, but we also consider odd lattices in these notes, so the resulting VOA can be a VOSA to be precise (see e.g. [DL93, Remark 12.38]). However, this distinction is not important for our purposes, so we will be unconcerned about that point. In these notes, we always assume that lattices are integral. We also always assume that lattices are Euclidean (positive-definite), and hence the resulting lattice VOA formulates a chiral CFT in the language of physics. As for Lorentzian lattice VOAs or more general VOAs to describe non-chiral CFTs (full CFTs in other words), see [HK05, Mor20, SS23].

The first step to constructing the lattice VOA V_L from a given lattice L is to furnish the lattice L with a cocycle factor $\hat{\varepsilon}$. We will eventually use the specific cocycle factor ε required by physics as in (5.34), but for a moment, let $\hat{\varepsilon} : L \times L \to \mathbb{Z}_2$ denote a general 2-cocycle to keep generality.

We will use some basic concepts of group extensions throughout the rest of this Section 5. See Appendix A for some basic facts on group extensions, although the description in this Section 5 is intended to be as self-contained as possible.

Let $\hat{L}$ be the central extension[20] of the lattice (as a free abelian group) L by $\mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1 \rangle$

$$1 \to \mathbb{Z}_2 \to \hat{L} \xrightarrow{-} L \to 0, \tag{5.36}$$

specified by a 2-cocycle $\hat{\varepsilon} : L \times L \to \mathbb{Z}_2$. In other words, $\hat{L}$ is $\mathbb{Z}_2 \times L$ as a set, and if we write its element as $\kappa^m e^k$ ($\kappa^m \in \mathbb{Z}_2, k \in L$), then $\hat{L}$ is a group specified by the multiplication

$$\kappa^m e^k \cdot \kappa^{m'} e^{k'} = \hat{\varepsilon}(k, k') \kappa^{m+m'} e^{k+k'}. \tag{5.37}$$

So, $e^k \in \hat{L}$ and $\hat{\varepsilon}$ correspond to $c_k(p)$ and ε in (5.34) respectively, under $\kappa = e^{\sqrt{-1}\pi}$.

Here is one comment on the normalization of the 2-cocycle. It follows from the 2-cocycle condition

$$\hat{\varepsilon}(k, k')\hat{\varepsilon}(k + k', k'') = \hat{\varepsilon}(k, k' + k'')\hat{\varepsilon}(k', k''), \tag{5.38}$$

that any 2-cocycle $\hat{\varepsilon} : L \times L \to \mathbb{Z}_2$ satisfies

$$\hat{\varepsilon}(k, 0) = \hat{\varepsilon}(0, k) = \hat{\varepsilon}(0, 0) \quad \text{for any } k \in L. \tag{5.39}$$

Furthermore, it is known that there exists a 2-cocycle $\hat{\varepsilon}$ satisfying the normalization condition

$$\hat{\varepsilon}(0, 0) = \kappa^0, \tag{5.40}$$

in any cohomology class in $H^2(L, \mathbb{Z}_2)$, and it defines an equivalent extension $\hat{L}$ to any 2-cocycle in the same cohomology class (see the last paragraph of Section A.1). Therefore, we will always assume that $\hat{\varepsilon}$ is a normalized one as in (5.40), without loss of generality. Then, we can observe that the multiplication of $\hat{L}$ is well-behaved in the sense that

$$\kappa^m e^0 \cdot \kappa^{m'} e^k = \kappa^{m+m'} e^k, \tag{5.41}$$

$$\kappa^m e^k \cdot \kappa^{m'} e^0 = \kappa^{m+m'} e^k, \tag{5.42}$$

and hence there is no confusion if we just write κ^m instead of $\kappa^m e^0$. We will also just write e^k instead of $\kappa^0 e^k$.

Now $\hat{L}$ has the multiplication structure reflecting the OPE (5.34), but not yet the structure of a $\mathbb{C}$-vector space of states. So we would like to consider the group algebra $\mathbb{C}[\hat{L}]$ of $\hat{L}$ over $\mathbb{C}$, but we have to identify κ with $e^{\sqrt{-1}\pi} \in \mathbb{C}$, in order to properly make $\kappa^m e^k$ and $\kappa^{m'} e^k$ be $\mathbb{C}$-linearly dependent. This is mathematically done as follows. We consider a representation $\mathbb{C}$ of the subgroup $\mathbb{Z}_2$ of $\hat{L}$, where the generator κ acts on $\mathbb{C}$ as multiplication by $e^{\sqrt{-1}\pi}$. We define

[20] More generally, [FLM88] deals with the central extension $\hat{L}$ of L by $\mathbb{Z}_s = \langle \kappa \mid \kappa^s = 1 \rangle$. Furthermore, we would like to consider the central extension by $\mathrm{U}(1) \cong \mathbb{R}/2\mathbb{Z} = \langle \kappa \rangle_{\mathbb{R}} / \langle \kappa \rangle_{2\mathbb{Z}}$ later. Therefore, the discussions below avoid using special properties for $\mathbb{Z}_2$, such as $\hat{\varepsilon}(k, k')^{-1} = \hat{\varepsilon}(k, k')$.

the $\hat{L}$-module $\mathbb{C}\{L\}$ as the induced representation of $\hat{L}$ from it, or equivalently, the extension of scalars of the module $\mathbb{C}$ from $\mathbb{C}[\mathbb{Z}_2]$ to $\mathbb{C}[\hat{L}]$:

$$\mathbb{C}\{L\} := \mathrm{Ind}^{\hat{L}}_{\mathbb{Z}_2 \text{ or } \mathrm{U}(1)}\mathbb{C} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\mathbb{Z}_2]} \mathbb{C}. \tag{5.43}$$

Roughly speaking, $\mathbb{C}\{L\}$ is just a group algebra $\mathbb{C}[\hat{L}]$ with $\kappa^m \cdot e^k = e^{\sqrt{-1}\pi m} e^k$.

The second step is to introduce the creation-annihilation operators. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ be the abelian Lie algebra, and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be its affine Lie algebra with the Lie bracket

$$[\alpha(m), \alpha'(m')] = \langle \alpha, \alpha' \rangle m \delta_{m+m',0} K, \tag{5.44}$$

where $\alpha(m) := \alpha \otimes t^m$ with $\alpha \in \mathfrak{h}$, $m \in \mathbb{Z}$, and the symmetric bilinear form $\langle -, - \rangle$ on L is extended to $\mathfrak{h}$ by $\mathbb{C}$-linearity.

Let $U(\hat{\mathfrak{h}})$ be the universal enveloping algebra of $\hat{\mathfrak{h}}$ (the algebra where $\alpha(m)\alpha'(m') - \alpha'(m')\alpha(m) = [\alpha(m), \alpha'(m')]$ holds), and define the $\hat{\mathfrak{h}}$-module $M(1)$ as

$$M(1) := U(\hat{\mathfrak{h}}) \otimes_{\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K} \mathbb{C}, \tag{5.45}$$

where $\mathfrak{h} \otimes \mathbb{C}[t]$ acts on $\mathbb{C}$ as $\alpha(m) \cdot \mathbb{C} = 0$ $(m \geq 0)$,[21] and K acts on $\mathbb{C}$ as multiplication by 1. As a vector space, $M(1)$ is isomorphic to $U(\hat{\mathfrak{h}}^-)$, which is the universal enveloping algebra of the abelian subalgebra

$$\hat{\mathfrak{h}}^- := \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}], \tag{5.46}$$

of $\hat{\mathfrak{h}}$.

Finally, the *lattice VOA* V_L is defined as the $\mathbb{C}$ vector space

$$V_L := M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\}, \tag{5.47}$$

with the VOA structure such as the vacuum vector and the state-field correspondence, which we do not write down here. Any element of V_L therefore can be written as a $\mathbb{C}$-linear combination of elements in the form of

$$\alpha_1(-m_1) \cdots \alpha_l(-m_l) e^k, \tag{5.48}$$

where $\alpha_1, \ldots, \alpha_l \in \mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, $m_1, \ldots, m_l \in \mathbb{Z}_{>0}$, and $k \in L$.

Let $e_1, \ldots, e_n$ be an integral basis of L. The element $e_{i_1}(-m_1) \cdots e_{i_l}(-m_l) e^k$ of V_L is usually denoted by $\alpha^{i_1}_{-m_1} \cdots \alpha^{i_l}_{-m_l} |k\rangle$ in physics literature, where the commutation relation of the creation-annihilation operators α^i_m is $[\alpha^i_m, \alpha^{i'}_{m'}] = \langle e_i, e_{i'} \rangle m \delta_{m+m',0}$. We also often use the creation-annihilation operators $\mathbb{a}^i_m := \mathbb{e}_i(m)$ ($m \neq 0$; see footnote 21) with respect to an orthonormal

[21] Here, $\alpha(0)$ also annihilates the states, unlike the momentum operators usually denoted by α^i_0 in physics as in the previous Section 5.2.1. This is just a problem of conventions.

basis $\mathbb{e}_1, \ldots, \mathbb{e}_n$ of $\mathbb{R}^n$ where the lattice L is embedded. Their commutation relation is of course $[\alpha^i_m, \alpha^{i'}_{m'}] = \delta_{i,i'} m \delta_{m+m',0}$.

The vacuum vector $\mathbf{1}$ of the lattice VOA V_L is $\mathbf{1} = e^0$. The Virasoro element ω is given by $\omega = \frac{1}{2}\sum_i \mathbb{e}_i(-1)^2\mathbf{1}$, and the corresponding operator is $T(z) = -\frac{1}{2}\sum_i :(\partial X^i(z))^2:$. Then the weight of the element (5.48), namely the eigenvalue with respect to L_0, can be calculated as

$$\mathrm{wt}(\alpha_1(-m_1)\cdots\alpha_l(-m_l)e^k) = m_1 + \cdots + m_l + \frac{1}{2}|k|^2, \tag{5.49}$$

and this weight introduces the grading of the VOA V_L.

5.2.3 Automorphism Group of Lattice VOA

The group $O(\hat{L})$

Since the lattice VOA V_L is built on the central extension $\hat{L}$ of the lattice L, before directly getting into the automorphism group of V_L, it is useful to first investigate the automorphism group of $\hat{L}$.

We define the *commutator map* $\hat{c} : L \times L \to \mathbb{Z}_2$ by

$$\hat{c}(k, k') := \hat{\varepsilon}(k, k')\hat{\varepsilon}(k', k)^{-1}. \tag{5.50}$$

If $\hat{\varepsilon}$ here is the specific one ε in (5.34), then it immediately follows from (5.34) that ε must satisfy

$$\varepsilon(k, k') = (-1)^{k\cdot k' + |k|^2|k'|^2}\varepsilon(k', k), \tag{5.51}$$

so we have $\hat{c}(k, k') = \kappa^{k\cdot k' + |k|^2|k'|^2}\;|_{\kappa=-1}$.

Forgetting the symmetric bilinear form on L, for a moment, we focus on the automorphism group $\mathrm{Aut}(L)$ of just a free abelian group L, instead of the isometry group $O(L)$ of the lattice L. The proposition [FLM88, Prop. 5.4.1] states that

$$1 \to \mathrm{Hom}(L, \mathbb{Z}_2) \xrightarrow{\sim} \mathrm{Aut}(\hat{L}, \kappa) \xrightarrow{-} \mathrm{Aut}(L, \hat{c}) \to 1 \tag{5.52}$$

is exact. (The proof is also reviewed in Appendix A.4.) The details of (5.52) are as follows. $\mathrm{Hom}(L, \mathbb{Z}_2) \xrightarrow{\sim} \mathrm{Aut}(\hat{L}, \kappa)$ maps $\eta \in \mathrm{Hom}(L, \mathbb{Z}_2)$ to

$$\tilde{\eta} : \qquad \hat{L} \to \hat{L} \tag{5.53}$$

$$\kappa^m e^k \mapsto \eta(k)\kappa^m e^k. \tag{5.54}$$

Note that $\eta \in \mathrm{Hom}(L, \mathbb{Z}_2)$ is determined only from the values of $\eta(e_1), \ldots, \eta(e_n)$, where $e_1, \ldots, e_n$ is a basis of L, and hence

$$\mathrm{Hom}(L, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^n. \tag{5.55}$$

$\mathrm{Aut}(\hat{L}, \kappa)$ is defined as

$$\mathrm{Aut}(\hat{L}, \kappa) := \{f \in \mathrm{Aut}(\hat{L}) \mid f(\kappa) = \kappa\}, \tag{5.56}$$

and $\mathrm{Aut}(L, \hat{c})$ is defined as

$$\mathrm{Aut}(L, \hat{c}) := \{g \in \mathrm{Aut}(L) \mid \hat{c}(g(k), g(k')) = \hat{c}(k, k')\}. \tag{5.57}$$

$\mathrm{Aut}(\hat{L}, \kappa) \twoheadrightarrow \mathrm{Aut}(L, \hat{c})$ maps $f \in \mathrm{Aut}(\hat{L})$ to

$$\bar{f} : L \to L \tag{5.58}$$
$$k \mapsto \overline{f(e^k)}, \tag{5.59}$$

where the natural projection $\hat{L} \twoheadrightarrow L$ of (5.36) is used.

Since 1 and κ are the only elements of finite order in $\hat{L}$, it follows that any $f \in \mathrm{Aut}(\hat{L})$ satisfies $f(\kappa) = \kappa$, and hence $\mathrm{Aut}(\hat{L}, \kappa)$ reduces to $\mathrm{Aut}(\hat{L})$ in the case at hand. However, this is the special property for the extension by $\mathbb{Z}_{s=2}$. If we consider a more general extension, say by $\mathbb{Z}_{s>2}$, then we cannot reduce $\mathrm{Aut}(\hat{L}, \kappa)$ to $\mathrm{Aut}(\hat{L})$ (see footnote 20).

Let us recall that L is a lattice, more than just a free abelian group, and move on to the isometry group $O(L)$ of the lattice from the automorphism group $\mathrm{Aut}(L)$ of the free abelian group. If the commutator map $\hat{c}$ depends only on the bilinear form of the lattice L, say $\hat{c}(k, k') = \kappa^{k \cdot k' + |k|^2 |k'|^2}$, then the isometry group $O(L)$ is a subgroup of $\mathrm{Aut}(L, \hat{c})$ defined in (5.57). In addition, if we define

$$O(\hat{L}) := \{f \in \mathrm{Aut}(\hat{L}, \kappa) \mid \bar{f} \in O(L)\}, \tag{5.60}$$

then we obtain the exact sequence [FLM88, Prop. 6.4.1]

$$1 \to \mathrm{Hom}(L, \mathbb{Z}_2) \xrightarrow{\tilde{}} O(\hat{L}) \xrightarrow{\bar{}} O(L) \to 1 \tag{5.61}$$

from (5.52).

Remark 5.1. For any $g \in O(L)$, it is known that there exists a lift $\hat{g} \in O(\hat{L})$ of g such that $\hat{g}(e^k) = e^k$ for any $k \in L$ fixed by g as $g(k) = k$ [Lep85, §5]. Such lift $\hat{g}$ is called the *standard lift* of g, and is sometimes of use in research. See for example [Bor92, Lemma 12.1], [Mö16, §5.3], [vEMS20, §7] for some properties of the standard lift. *(Remark ends.)*

The automorphism group $\mathrm{Aut}(V_L)$

Following [FLM88, §8.10] or [DN99, §2.3], the definition of an automorphism of a VOA is as follows.

Definition 5.2 (automorphism of a VOA)**.** An *automorphism* of a VOA V over $\mathbb{C}$ is a map $F : V \to V$ such that

(1) it is a $\mathbb{C}$-linear automorphism on the $\mathbb{C}$-vector space V.

(2) it preserves all the products.[22] More precisely, $F(v_{(m)}(v')) = (F(v))_{(m)}(F(v'))$ for any $v, v' \in V$ and $m \in \mathbb{Z}$. Or equivalently, $F \circ Y(v,z) \circ F^{-1} = Y(F(v), z)$ for any $v \in V$.

(3) it preserves the Virasoro element $F(\omega) = \omega$, and hence preserves the grading of V.

■

Any element f of $O(\hat{L})$ induces an automorphism F of a lattice VOA V_L, which acts on the state (5.29) as

$$F(\alpha_1(-m_1)\cdots\alpha_l(-m_l)e^k) = \bar{f}(\alpha_1)(-m_1)\cdots\bar{f}(\alpha_l)(-m_l)f(e^k), \tag{5.63}$$

where $\bar{f} \in O(L)$ was defined in (5.59) and extended by $\mathbb{C}$-linearity here. This preserves the group structure of $O(\hat{L})$, and hence $O(\hat{L}) \cong (\mathbb{Z}_2)^n.O(L)$ can be regarded as a subgroup of the automorphism group $\mathrm{Aut}(V_L)$ of the lattice VOA V_L. The whole $\mathrm{Aut}(V_L)$ is determined in [DN99] as follows.

Theorem 5.3 ([DN99, Theorem 2.1])**.** *For a positive-definite even lattice L, the automorphism group $\mathrm{Aut}(V_L)$ is generated by $O(\hat{L})$ and N defined as*

$$N := \{\exp(v_{(0)}) \mid v \in V_L, \mathrm{wt}(v) = 1\}. \tag{5.64}$$

In short,

$$\mathrm{Aut}(V_L) = N \cdot O(\hat{L}). \tag{5.65}$$

N is a normal subgroup of $\mathrm{Aut}(V_L)$, and $N \cap O(\hat{L})$ contains $\mathrm{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$.

Remark 5.4. We will not need the details of the subgroup $N \subset \mathrm{Aut}(V_L)$ below, but here we provide some explanation in this Remark.

An *(even) derivation*[23] of a VOA V is a linear endomorphism $D : V \to V$ such that

- $D(v_{(m)}(v')) = (D(v))_{(m)}(v') + v_{(m)}(D(v'))$ for any $v, v' \in V$ and $m \in \mathbb{Z}$,
- $D(\omega) = 0$.

[22] A vertex algebra V in the sense of Definition 4.2 satisfies the *Borcherds OPE formula* [Kac98, Thm. 4.6]

$$Y(v,z_1)Y(w,z_2) = \sum_{n=0}^{\infty} \frac{Y(v_{(n)}w, z_2)}{(z_1 - z_2)^{n+1}} + :Y(v,z_1)Y(w,z_2):, \tag{5.62}$$

in the domain $|z_1| > |z_2|$ for any $v, w \in V$. In view of this, the condition (2) of Definition 5.2 can be phrased as "preserve OPE".

[23] An *odd derivation* [Kac98, §4.3] requires $D(v_{(m)}(v')) = (D(v))_{(m)}(v') + (-1)^{|v|}v_{(m)}(D(v'))$ instead of the first equation, but we do not use it in these notes.

Combining these equations and $L_0 = \omega_{(1)}$, we can see $D(L_0 v) = L_0(Dv)$, and hence a derivation preserves the grading of V. We can easily see that $\exp(D)$ is an automorphism of V.

For any element $u \in V$ of weight 1, $u_{(0)}$ is a derivation as follows. The first equation follows from $[u_{(0)}, v_{(m)}] = (u_{(0)}v)_{(m)}$ [Kac98, Eq. (4.6.3)]. To see the second equation, we have $[u_{(0)}, L_{m'}] = 0$ from $[u_{(m)}, L_{m'}] = (m - (\mathrm{wt}(u) - 1)m')u_{(m+m')}$ [Kac98, Cor. 4.10 (iii)], and hence $u_{(0)}(\omega) = u_{(0)}L_{-2}\mathbf{1} = L_{-2}u_{(0)}\mathbf{1} = 0$, using the Definition 4.1 of the vacuum vector $\mathbf{1}$.

Therefore, $\exp(u_{(0)})$ with $\mathrm{wt}(u) = 1$ is an automorphism of V, called an *inner automorphism* in [Kac98, Remark 4.9c]. So

$$N := \{\exp(u_{(0)}) \mid u \in V, \mathrm{wt}(u) = 1\} \tag{5.66}$$

is indeed a subgroup of $\mathrm{Aut}(V_L)$. Furthermore, N is a normal subgroup, because any $F \in \mathrm{Aut}(V)$ satisfies $F \circ \exp(u_{(0)}) \circ F^{-1} = \exp((F(u))_{(0)})$ and $\mathrm{wt}(F(u)) = \mathrm{wt}(u) = 1$.

For example, in the case of a lattice VOA V_L, if we take the weight-1 element u as $e_i(-1)e^0$, or $\alpha^i_{-1}|0\rangle$ in physics notation, then $u_{(0)}$ is the momentum operator α^i_0 described in Section 5.2.1 (see also footnote 21), which corresponds to $\langle e_i, -\rangle \in \mathrm{Hom}(L, \mathbb{R})$. Recalling $\omega = \frac{1}{2}\sum_i \mathbb{e}_i(-1)^2 e^0$, we can explicitly see that $\alpha^i_0(\omega) = 0$. We can also see that $\exp(t\alpha^i_0) \in N$ is in $\mathrm{Hom}(L, \mathbb{Z}_2)$ for appropriate $t \in \mathbb{C}$, and this is the basic argument to show that $N \cap O(\hat{L})$ contains $\mathrm{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$.

If the lattice L contains a vector k of squared length 2, then $e^k \in V_L$ is also a weight-1 element. On the other hand, if L does not have a vector of squared length 2, then $N \cap O(\hat{L})$ coincides with $\mathrm{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$, and $\mathrm{Aut}(V_L) = N.O(L)$ [Lam20, Remark 2.3]. The Leech lattice Λ_{24} is such an example. *(Remark ends.)*

As a study of the structure of $\mathrm{Aut}(V_L)$, we can consider the following question: does the isometry group $O(L)$ of the lattice L lift to a subgroup of the automorphism group $\mathrm{Aut}(V_L)$ of the lattice VOA V_L? Now that we have seen that $O(\hat{L})$ is a subgroup of $\mathrm{Aut}(V_L)$, this question is equivalent to whether the group extension (5.61) splits or not. This is the main subject of [Oka24].

5.3 $\mathbb{Z}_2$-Orbifold of Lattice VOA

When a CFT $\mathcal{T}$ has a finite group symmetry $G \subset \mathrm{Aut}(\mathcal{T})$ and it is non-anomalous, then we can construct a new CFT $\mathcal{T}/G$ consisting of G-invariant states, which is called the orbifold of $\mathcal{T}$ by G. The monster VOA $V^\natural$ is the orbifold of the Leech lattice VOA $V_{\Lambda_{24}}$ by the reflection $\mathbb{Z}_2$ symmetry $X(z) \mapsto -X(z)$. So we review the orbifold $\tilde{V}_L$ of a lattice VOA V_L by the reflection $\mathbb{Z}_2$ symmetry in Section 5.3.1, and the automorphisms of the resulting theory $\tilde{V}_L$ in Section 5.3.2. A lattice VOA has another $\mathbb{Z}_2$ symmetry, the shift $\mathbb{Z}_2$ symmetry $X(z) \mapsto X(z) + \pi\chi$, and we will review it and the relations between two orbifolds by the reflection $\mathbb{Z}_2$ and by the shift $\mathbb{Z}_2$ in Section 5.3.3. This perspective of another symmetry was not explicitly used in the original proof of $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$ by [FLM88], but [DGM90a] revealed that it provides a clear way of understanding the proof. Lastly, in Section 5.3.4, we will briefly review a uniform treatment of $\mathbb{Z}_2$-orbifold and fermionization, although it is not directly related to the monstrous moonshine.

5.3.1 Orbifold by Reflection $\mathbb{Z}_2$ Symmetry

The reflection automorphism θ_0

We first explain what the reflection $\mathbb{Z}_2$ symmetry of a lattice VOA V_L is.

Proposition 5.5. *Any lift $\theta \in O(\hat{L})$ of $-\mathrm{id}_L \in O(L)$ in the exact sequence (5.61) satisfies $\theta^2 = \mathrm{id}_{\hat{L}}$.*

Proof. Since $e^k \cdot e^{-k} = \hat{\varepsilon}(k,-k)$, we have $e^{-k} = \hat{\varepsilon}(k,-k)(e^k)^{-1}$. Hence, for a given $k \in L$, there is $m \in \mathbb{Z}$ such that $\theta(e^k) = \kappa^m (e^k)^{-1}$. Therefore, by using $\theta(\kappa) = \kappa$ from the definition (5.60) of $O(\hat{L})$,

$$\theta^2(e^k) = \kappa^m \theta(e^k)^{-1} = \kappa^m \kappa^{-m} e^k = e^k. \tag{5.67}$$

□

Suppose the 2-cocycle $\hat{\varepsilon} : L \times L \to \mathbb{Z}_2$ is bihomomorphic, or we customarily say it is bilinear. In fact, we will eventually use the bilinear one (C.13) for the 2-cocycle ε appearing in (5.34). Then, the map $\theta_0 : \hat{L} \to \hat{L}$ defined as

$$\theta_0(\kappa^m e^k) = \kappa^m e^{-k} \tag{5.68}$$

is a homomorphism, because

$$\theta_0(e^k \cdot e^l) = \hat{\varepsilon}(k,l) e^{-k-l}, \quad \theta_0(e^k) \cdot \theta_0(e^l) = \hat{\varepsilon}(-k,-l) e^{-k-l}, \tag{5.69}$$

and $\hat{\varepsilon}(k,l) = \hat{\varepsilon}(-k,-l)$ by the assumption of bilinearity. θ_0 is obviously an automorphism in $O(\hat{L})$, and hence it extends to the automorphism of V_L as in (5.63):

$$\theta_0(\alpha_1(-m_1) \cdots \alpha_l(-m_l) e^k) = (-\alpha_1)(-m_1) \cdots (-\alpha_l)(-m_l) e^{-k}. \tag{5.70}$$

By Proposition 5.5, this θ_0 generates a subgroup $\mathbb{Z}_2$ of $\mathrm{Aut}(V_L)$, and this is the *reflection $\mathbb{Z}_2$ symmetry* of the lattice VOA V_L.

In physics notation, the action of θ_0 on a state is

$$\theta_0 \alpha^{i_1}_{-m_1} \cdots \alpha^{i_l}_{-m_l} |k\rangle = (-\alpha^{i_1}_{-m_1}) \cdots (-\alpha^{i_l}_{-m_l}) |-k\rangle, \tag{5.71}$$

and this can be described as the reflection symmetry of the chiral bosons (5.31)

$$X^i(z) \mapsto -X^i(z) \quad \text{for all } i = 1, \ldots, n. \tag{5.72}$$

The θ_0-twisted sector

The orbifold of a lattice VOA V_L by the reflection $\mathbb{Z}_2$ symmetry $\mathbb{Z}_2 = \langle \theta_0 \rangle \subset \mathrm{Aut}(V_L)$ is done by the following two steps.

1. Introduce the θ_0-twisted sector $(V_L)_{\mathrm{tw}}$.

2. Project the whole $V_L \oplus (V_L)_{\mathrm{tw}}$ to the θ_0-invariant states $(V_L)^0 \oplus (V_L)^0_{\mathrm{tw}} =: \tilde{V}_L$.

The sectors[24] can be summarized as

$$\begin{array}{c|cc} \mathbb{Z}_2 = \langle\theta_0\rangle & \text{untwisted} & \text{twisted} \\ \hline \text{even} & (V_L)^0 & (V_L)^0_{\mathrm{tw}} \\ \text{odd} & (V_L)^1 & (V_L)^1_{\mathrm{tw}} \end{array} . \tag{5.73}$$

The construction of an orbifold by more general finite group symmetry is also accomplished by similar two steps, as reviewed in Section F.2.1.

We review the construction of the twisted sector $(V_L)_{\mathrm{tw}}$. Let L be an even self-dual lattice of rank n, and the 2-cocycle $\hat{\varepsilon}$ be the specific one ε in (C.13) associated with the commutator $c(k,k') = \kappa^{k\cdot k'}$ and the quadratic form $q(k) = \kappa^{\frac{1}{2}|k|^2}$, in the rest of this Section 5.3. As a result, the action of θ_0 can be written as

$$\theta_0(e^k) = e^{-k} = \varepsilon(k,-k)(e^k)^{-1} = \kappa^{\frac{1}{2}|k|^2}(e^k)^{-1}, \tag{5.74}$$

where in the last equation, we used the bilinearity of ε and Lemma A.4 (2) $\varepsilon(k,k) = q(k) = \kappa^{\frac{1}{2}|k|^2}$.

If we follow [FLM88], we first define

$$K := \{\theta_0(a)^{-1}a \mid a \in \hat{L}\} \tag{5.75}$$

$$= \{\kappa^{\frac{1}{2}|k|^2}(e^k)^2 \mid k \in L\} \tag{5.76}$$

$$= \{e^{2k} \mid k \in L\}. \tag{5.77}$$

According to Theorem A.7, K is a subgroup of the center of $\hat{L}$, and we have a central extension

$$1 \to \mathbb{Z}_2 \to \hat{L}/K \to L/2L \to 0. \tag{5.78}$$

Similarly to the notation $\kappa^m e^k$ of an element of $\hat{L}$, we may write an element of $\hat{L}/K$ as $\kappa^m\gamma_{k+2L}$. We will built the twisted sector on an irreducible representation of $Q = \hat{L}/K$.

A more physicist-friendly description of the group Q is introduced in [DGM94, §5.3], so let us follow it below. We define the *gamma matrix algebra* $\Gamma(L)$ *associated with* L as the unital (that is, $\Gamma(L)$ contains 1 and hence $\mathbb{C}$) $\mathbb{C}$-algebra generated by $\{\gamma_k\}_{k\in L}$ satisfying

$$\gamma_k\gamma_{k'} = (-1)^{k\cdot k'}\gamma_{k'}\gamma_k = \varepsilon(k,k')\gamma_{k+k'}, \quad \gamma_k^2 = (-1)^{\frac{1}{2}|k|^2}. \tag{5.79}$$

If we identify γ_k with e^k, then $\Gamma(L)$ is roughly the algebra $\mathbb{C}\{L\}$ with the relation $(e^k)^2 = (-1)^{\frac{1}{2}|k|^2}$ introduced, which corresponds to the quotient by K.

We define a group $Q := \{\pm\gamma_k \mid k \in L\}$. From the relations (5.79), it is obvious that $|Q| = 2^{1+n}$. In particular, if we take a $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of L, then

$$Q = \{\pm\gamma_{e_1}^{t_1}\cdots\gamma_{e_n}^{t_n}\}_{t_i=0,1} . \tag{5.80}$$

Moreover, Q is an extraspecial 2-group, which is defined as follows.

[24] $(V_L)^0$ and $(V_L)^0_{\mathrm{tw}}$ here in the main text are denoted by $(V_L)^{\theta_0}$ and $(V_L^T)^{\theta_0}$ respectively in [FLM88]. $(V_L)^0, (V_L)^1, (V_L)^0_{\mathrm{tw}}, (V_L)^1_{\mathrm{tw}}$, and $\tilde{V}_L$ here are denoted by $\mathcal{H}^+(L), \mathcal{H}^-(L), \mathcal{H}_T^+(L), \mathcal{H}_T^-(L)$, and $\tilde{\mathcal{H}}(L)$, respectively in [DGM94].

Definition 5.6 (extraspecial p-group)**.** A group G is called a *p-group*, if its order $|G|$ is a power of a prime p. A group G is called an *elementary abelian p-group*, if it is isomorphic to $(\mathbb{Z}_p)^n$ for some $n \in \mathbb{Z}_{>0}$. A group G is called an *extraspecial p-group* and denoted by p^{1+n}, if the center $Z(G)$ of G satisfies $Z(G) \cong \mathbb{Z}_p$ and $G/Z(G) \cong (\mathbb{Z}_p)^n$. ■

It is known that for any extraspecial p-group p^{1+n}, n is even.

Proposition 5.7. *Q is an extraspecial 2-group 2^{1+n}.*

Proof. We can show that the center of Q is $\{\pm 1\}$ as follows. If γ_k is in the center of Q, then $k \cdot k' \in 2\mathbb{Z}$ for any $k' \in L$. This means $\frac{k}{2} \in L^*$, where L^* is the dual lattice defined as (2.14). Since we assumed L is self-dual, $k \in 2L$, and therefore $\gamma_k = \pm 1$ by using the relations (5.79).

Finally, $Q/\{\pm 1\} \cong L/2L \cong (\mathbb{Z}_2)^n$. □

The irreducible representations of Q are, the 2^n one-dimensional ones of $Q/\{\pm 1\} \cong (\mathbb{Z}_2)^n$ extended so that they map the center $\{\pm 1\}$ to 1, and the unique $2^{\frac{n}{2}}$-dimensional faithful one. We can understand this $2^{\frac{n}{2}}$-dimensional irreducible representation as follows. We can show [DGM90b, Appendix C] that Q can be written as

$$Q = \{\pm \tilde{\gamma}_1^{t_1} \cdots \tilde{\gamma}_n^{t_n}\}_{t_i = 0,1}\,, \tag{5.81}$$

using the usual gamma matrices (the Clifford algebra)

$$\tilde{\gamma}_i \tilde{\gamma}_j = -\tilde{\gamma}_j \tilde{\gamma}_i \; (i \neq j), \quad \tilde{\gamma}_i^2 = \tilde{\varepsilon}_i, \tag{5.82}$$

or equivalently $\{\tilde{\gamma}_i, \tilde{\gamma}_j\} = 2\tilde{\varepsilon}_i \delta_{ij}$, for some $\tilde{\varepsilon}_i = \pm 1$. Then, the construction of the $2^{\frac{n}{2}}$-dimensional irreducible representation can be done by the usual argument of representing $\tilde{\gamma}_i$ as the tensor product of $\frac{n}{2}$ Pauli matrices.

Among the irreducible representations of Q, this $2^{\frac{n}{2}}$-dimensional one is the only one which can extend to the representation of the whole algebra $\Gamma(L)$. (The other one-dimensional ones cannot satisfy for example $\{\tilde{\gamma}_i, \tilde{\gamma}_j\} = 0$.) Therefore, the gamma matrix algebra $\Gamma(L)$ have the unique irreducible representation, which we will write $\mathcal{X}(L)$, of dimension $2^{\frac{n}{2}}$.

Remark 5.8. For more general even lattices L, the center of the group $\{\pm \gamma_k \mid k \in L\}$ can be bigger than $\{\pm 1\}$. We still can construct its irreducible representations by a similar discussion using the usual gamma matrices, but the resulting representations are not unique in general, because there are degrees of freedom of assigning scalers $\pm 1, \pm\sqrt{-1}$ to the central elements. Such irreducible representations have a common dimension smaller than $2^{\frac{n}{2}}$ in general. See [DGM90b, Appendix C] for more details. *(Remark ends.)*

We introduce the creation-annihilation operators $c(r)$ ($c \in L \otimes_{\mathbb{Z}} \mathbb{C}$, $r \in \mathbb{Z} + \frac{1}{2}$) satisfy

$$[c(r), c'(r')] = \langle c, c' \rangle r \delta_{r+r',0}. \tag{5.83}$$

Then the θ_0-twisted sector $(V_L)_{\mathrm{tw}}$ is constructed as the $\mathbb{C}$-vector space spanned by the elements in the form of

$$c_1(-r_1)\cdots c_l(-r_l)|s\rangle, \tag{5.84}$$

where $c_1,\ldots,c_l \in L\otimes_{\mathbb{Z}}\mathbb{C}$, $r_1,\ldots,r_l \in (\mathbb{Z}+\frac{1}{2})_{>0}$, and $|s\rangle \in \mathcal{X}(L)$. We also often use the creation-annihilation operators $\mathbb{c}_r^i := \mathbb{e}_i(r)$ with respect to an orthonormal basis $\mathbb{e}_1,\ldots,\mathbb{e}_n$ of $\mathbb{R}^n$ where the lattice L is embedded.

The operators

$$L_m = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} :\mathbb{c}_r\mathbb{c}_{m-r}: + \frac{n}{16}\delta_{m,0} \tag{5.85}$$

satisfy the Virasoro algebra (4.12) of central charge n. Then the state (5.84) is an eigenstate of L_0 with eigenvalue

$$r_1+\cdots+r_l+\frac{n}{16}. \tag{5.86}$$

Projection to θ_0-invariant states

To complete the construction of the orbifold, we project the whole $V_L\oplus(V_L)_{\mathrm{tw}}$ to the θ_0-invariant states $(V_L)^0\oplus(V_L)^0_{\mathrm{tw}} =: \tilde{V}_L$.

Recall that θ_0 acts on V_L as in (5.71)

$$\theta_0\alpha_m^i\theta_0^{-1} = -\alpha_m^i, \quad \theta_0|k\rangle = |-k\rangle. \tag{5.87}$$

Therefore, the θ_0-invariant (θ_0-even) sector $(V_L)^0$ and the θ_0-odd sector $(V_L)^1$ are

$$\begin{aligned}(V_L)^0 = \mathrm{Span}_{\mathbb{C}}(\{&(\text{even number of } \alpha_m^i\text{'s})(|k\rangle+|-k\rangle)\}\\ &\oplus\{(\text{odd number of } \alpha_m^i\text{'s})(|k\rangle-|-k\rangle)\}),\end{aligned} \tag{5.88}$$

$$\begin{aligned}(V_L)^1 = \mathrm{Span}_{\mathbb{C}}(\{&(\text{odd number of } \alpha_m^i\text{'s})(|k\rangle+|-k\rangle)\}\\ &\oplus\{(\text{even number of } \alpha_m^i\text{'s})(|k\rangle-|-k\rangle)\}).\end{aligned} \tag{5.89}$$

We extend the action of θ_0 on V_L to the action of $\hat{\theta}_0$ on $V_L\oplus(V_L)_{\mathrm{tw}}$ by defining

$$\hat{\theta}_0\mathbb{c}_r^i\hat{\theta}_0^{-1} = -\mathbb{c}_r^i, \quad \hat{\theta}_0|s\rangle = (-1)^{\frac{n}{8}}|s\rangle. \tag{5.90}$$

Recall that the rank n of an even self-dual lattice L is a multiple of 8. Then, the $\hat{\theta}_0$-even sector $(V_L)^0_{\mathrm{tw}}$ and the $\hat{\theta}_0$-odd sector $(V_L)^1_{\mathrm{tw}}$ are

$$(V_L)^0_{\mathrm{tw}} = \mathrm{Span}_{\mathbb{C}}\{\text{states in } (V_L)_{\mathrm{tw}} \text{ with an integer } L_0\text{-eigenvalue (5.86)}\}, \tag{5.91}$$

$$(V_L)^1_{\mathrm{tw}} = \mathrm{Span}_{\mathbb{C}}\{\text{states in } (V_L)_{\mathrm{tw}} \text{ with a half-integer } L_0\text{-eigenvalue (5.86)}\}. \tag{5.92}$$

We finally define the orbifold $\tilde{V}_L$ of the lattice VOA V_L by the reflection $\mathbb{Z}_2$ symmetry as

$$\tilde{V}_L := (V_L)^0\oplus(V_L)^0_{\mathrm{tw}}. \tag{5.93}$$

The results of [FLM88], [DGM94, Lemma 5.3, Theorem 5.4] show that

- $(V_L)^0$ is a sub-VOA of V_L. (This holds for a general Euclidean even lattice L.)
- $(V_L)^0_{\rm tw}$ is a module of $(V_L)^0$.
- $\tilde{V}_L = (V_L)^0 \oplus (V_L)^0_{\rm tw}$ has a structure of VOA, extended from that on $(V_L)^0$. (This holds for a general Euclidean even lattice such that $\sqrt{2}L^*$ is also even.)

From the last point, we can indeed call the L_0-eigenvalue (5.86) the weight.

The reflection $\mathbb{Z}_2$ orbifold $\tilde{V}_{\Lambda_{24}}$ of the Leech lattice VOA is called the *monster VOA* or the *moonshine module*, and denoted by $V^\natural$.

5.3.2 Automorphisms of Reflection $\mathbb{Z}_2$ Orbifold

As mentioned in Section 2.4, the monster group $\mathbb{M}$ is generated by a subgroup $C(\Lambda_{24}) = 2^{1+24}.\mathrm{Co}_1$ and a specific order-2 element σ. In the proof of $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$, the subgroup $C(\Lambda_{24})$ is coming from automorphisms acting on $(V_{\Lambda_{24}})^0$ and $(V_{\Lambda_{24}})^0_{\rm tw}$ separately. In this Section 5.3.2, we will review these automorphisms, following [FLM88, §10.4] and [DGM94, §6.1].

The group $O(\hat{L})/\langle\theta_0\rangle$ acting on the untwisted θ_0-invariant sector $(V_L)^0$

Let us begin with the automorphisms of the untwisted θ_0-invariant sector $(V_L)^0$. Recall that V_L was built on $\hat{L}$, and $O(\hat{L})$ is a subgroup of $\mathrm{Aut}(V_L)$. Since $\theta_0 \in O(\hat{L})$ acts on $(V_L)^0$ trivially, $\mathrm{Aut}((V_L)^0)$ has a subgroup $O(\hat{L})/\langle\theta_0\rangle$. In the exact sequence (5.61), θ_0 is a lift of $-\mathrm{id}_L \in O(L)$, so we have

$$1 \to \mathrm{Hom}(L, \mathbb{Z}_2) \to O(\hat{L})/\langle\theta_0\rangle \to O(L)/\{\pm\mathrm{id}_L\} \to 1. \tag{5.94}$$

Since K defined in (5.75) is preserved by θ_0 as a subgroup of $\hat{L}$, the group $O(\hat{L})/\langle\theta_0\rangle$ is a subgroup of $\mathrm{Aut}(\hat{L}/K)$. Recall $Q = \hat{L}/K$. We define the natural group homomorphism

$$\varphi : O(\hat{L}) \to O(\hat{L})/\langle\theta_0\rangle \subset \mathrm{Aut}(Q). \tag{5.95}$$

The entire automorphism group $\mathrm{Aut}(Q)$ of the group Q can be described as follows. Recall that the group $Q = \hat{L}/K$ satisfies the exact sequence (5.78), similar to (5.36). As a result, similarly to (5.52), we have[25] the following exact sequence [FLM88, Prop. 5.4.5]

$$1 \to \mathrm{Hom}(L/2L, \mathbb{Z}_2) \xrightarrow{\tilde{}} \mathrm{Aut}(Q) \xrightarrow{\bar{}} \mathrm{Aut}(L/2L, q) \to 1, \tag{5.96}$$

where $q(k) = \kappa^{\frac{1}{2}|k|^2}|_{\kappa=-1}$ is the quadratic form and $\mathrm{Aut}(L/2L, q) := \{g \in \mathrm{Aut}(L/2L) \mid q(g(k)) = q(k)\}$. $\eta \in \mathrm{Hom}(L/2L, \mathbb{Z}_2)$ acts on $\gamma_k \in Q$ as $\tilde{\eta}(\gamma_k) = \eta(k)\gamma_k$, and $f \in \mathrm{Aut}(Q)$ maps $\gamma_k \in Q$ to $\pm\gamma_{\bar{f}(k)}$. Here, from the exact sequence (5.78), we regarded k of $\gamma_k \in Q$ as an element of $L/2L$.

[25] Precisely speaking, we first have to consider $\mathrm{Aut}(Q,\kappa) := \{f \in \mathrm{Aut}(Q) \mid f(\kappa) = \kappa\}$ similarly to (5.52), but in the case at hand, since the center of Q is $\{1,\kappa\}$, any automorphism of Q maps κ to κ, so we just have $\mathrm{Aut}(Q,\kappa) = \mathrm{Aut}(Q)$.

Since the quadratic form q only depends on the bilinear form of L, $\mathrm{Aut}(L/2L, q)$ contains $O(L)/\{\pm\mathrm{id}_L\}$ as a subgroup. By restricting the exact sequence (5.96) from $\mathrm{Aut}(L/2L, q)$ to $O(L)/\{\pm\mathrm{id}_L\}$, we obtain a subgroup of $\mathrm{Aut}(Q)$. The restricted exact sequence coincides with (5.94), and therefore this subgroup of $\mathrm{Aut}(Q)$ is exactly $O(\hat{L})/\langle\theta_0\rangle$.

The group $C_{\mathcal{X}(L)}$ acting on the Q-module $\mathcal{X}(L)$

Next, we consider a certain subgroup of the linear automorphism group $\mathrm{Aut}(\mathcal{X}(L))$ of $\mathcal{X}(L)$. Since $\mathcal{X}(L)$ is a Q-module, we have $Q \subset \mathrm{Aut}(\mathcal{X}(L))$. The normalizer[26] $N_{\mathrm{Aut}(\mathcal{X}(L))}(Q)$ of Q in $\mathrm{Aut}(\mathcal{X}(L))$ is defined as

$$N_{\mathrm{Aut}(\mathcal{X}(L))}(Q) := \{f \in \mathrm{Aut}(\mathcal{X}(L)) \mid fQ = Qf\} \tag{5.97}$$
$$= \{f \in \mathrm{Aut}(\mathcal{X}(L)) \mid f \bullet f^{-1} \in \mathrm{Aut}(Q)\}. \tag{5.98}$$

In other words, for any $\gamma \in Q$, if we write the action of $f \in \mathrm{Aut}(\mathcal{X}(L))$ as

$$f(\gamma|s\rangle) = f(\gamma)f|s\rangle \quad \text{for } |s\rangle \in \mathcal{X}(L), \tag{5.99}$$

then $f(\gamma) = f \circ \gamma \circ f^{-1}$ is again in Q if $f \in N_{\mathrm{Aut}(\mathcal{X}(L))}(Q)$. It is known [FLM88, Prop. 5.5.3] that we have the following exact sequence

$$1 \to \{\pm\mathrm{id}_{\mathcal{X}(L)}\} \to N_{\mathrm{Aut}(\mathcal{X}(L))}(Q) \xrightarrow{f \,\mapsto\, f\bullet f^{-1}} \mathrm{Aut}(Q) \to 1. \tag{5.100}$$

When we extend an automorphism on $\mathcal{X}(L)$ to that on $(V_L)_{\mathrm{tw}}$, we will need $O(L)$ which acts on the creation-annihilation operators $c(r)$. In (5.100), the source of it is the subgroup $O(\hat{L})/\langle\theta_0\rangle$ of $\mathrm{Aut}(Q)$. We define a subgroup $C_{\mathcal{X}(L)}$ of $N_{\mathrm{Aut}(\mathcal{X}(L))}(Q)$ by restricting the exact sequence (5.100) as

$$1 \to \{\pm\mathrm{id}_{\mathcal{X}(L)}\} \to C_{\mathcal{X}(L)} \xrightarrow{f \,\mapsto\, f\bullet f^{-1}} O(\hat{L})/\langle\theta_0\rangle \to 1. \tag{5.101}$$

Note that, in addition to (5.100), $N_{\mathrm{Aut}(\mathcal{X}(L))}(Q)$ also satisfies the exact sequence

$$1 \to Q \to N_{\mathrm{Aut}(\mathcal{X}(L))}(Q) \xrightarrow{f \,\mapsto\, \overline{f\bullet f^{-1}}} \mathrm{Aut}(L/2L, q) \to 1, \tag{5.102}$$

and correspondingly,

$$1 \to Q \to C_{\mathcal{X}(L)} \xrightarrow{f \,\mapsto\, \overline{f\bullet f^{-1}}} O(L)/\{\pm\mathrm{id}_L\} \to 1. \tag{5.103}$$

(5.102) can be shown as follows. Combining (5.96) and (5.100), we have $N_{\mathrm{Aut}(\mathcal{X}(L))}(Q) \cong 2.\mathrm{Aut}(Q) \cong 2.(2^n.\mathrm{Aut}(L/2L, q))$, and the homomorphism $N_{\mathrm{Aut}(\mathcal{X}(L))}(Q) \to \mathrm{Aut}(L/2L, q); f \mapsto \overline{f \bullet f^{-1}}$. The kernel of this homomorphism obviously contains Q because $\gamma_k\gamma_{k'}\gamma_k^{-1} = \pm\gamma_{k'}$, and this group Q has order 2^{1+n}, which exhausts the order of the whole kernel. So we obtained the exact sequence (5.102).

[26]The *normalizer* of a subset H of a group G is defined as $N_G(H) := \{g \in G \mid gH = Hg\}$. If H is a subgroup of G, then $N_G(H)$ is the largest subgroup of G which contains H as a normal subgroup, hence its name.

The group $\hat{C}(L)$ acting on the whole $V_L \oplus (V_L)_{\text{tw}}$

In order to extend the action of $C_{\mathcal{X}(L)}$ on $\mathcal{X}(L)$ to the action on $(V_L)_{\text{tw}}$, we want the action of $g \in O(L)$ on the creation-annihilation operators $c(r) \mapsto g(c)(r)$, so we would like to revive $O(\hat{L})$ instead of $O(\hat{L})/\langle\theta_0\rangle$. This also incorporates the action of $O(\hat{L})$ on the untwisted sector V_L at the same time.

We define

$$\hat{C}(L) := \{(f, f_{\mathcal{X}(L)}) \in O(\hat{L}) \times C_{\mathcal{X}(L)} \mid \varphi(f) = f_{\mathcal{X}(L)} \bullet f_{\mathcal{X}(L)}^{-1} \in O(\hat{L})/\langle\theta_0\rangle\}. \tag{5.104}$$

It can be summarized into the following commutative diagram.

$$\begin{array}{ccccc} & & \hat{C}(L) = 2^{1+n}.O(L) & & \\ & \swarrow^{/\langle(\theta_0, \mathrm{id}_{\mathcal{X}(L)})\rangle} & & \searrow^{/\langle(\mathrm{id}_{\hat{L}}, -\mathrm{id}_{\mathcal{X}(L)})\rangle} & \\ C_{\mathcal{X}(L)} = 2^{1+n}.(O(L)/\{\pm\mathrm{id}_L\}) & & & & O(\hat{L}) = 2^n.O(L) \\ & \searrow_{f \mapsto f \bullet f^{-1}} & & \swarrow_{\varphi} & \\ & & O(\hat{L})/\langle\theta_0\rangle = 2^n.(O(L)/\{\pm\mathrm{id}_L\}) & & \end{array} \tag{5.105}$$

Here, we have the exact sequence

$$1 \to Q \xrightarrow{\gamma \mapsto (\gamma \bullet \gamma^{-1}, \gamma)} \hat{C}(L) \xrightarrow{(f, f_{\mathcal{X}(L)}) \mapsto \bar{f}} O(L) \to 1, \tag{5.106}$$

where $\gamma \bullet \gamma^{-1} \in \mathrm{Aut}(Q)$ in (5.100) is regarded as an element of $\mathrm{Hom}(L/2L, \mathbb{Z}_2) \subset \mathrm{Aut}(Q)$ as in (5.96), and extended to an element of $\mathrm{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$. In fact, $\gamma_k \gamma_{k'} \gamma_k^{-1} = \eta_k(k')\gamma_{k'}$ for some $\eta_k \in \mathrm{Hom}(L, \mathbb{Z}_2)$.

The action of $F = (f, f_{\mathcal{X}(L)}) \in \hat{C}(L)$ on V_L is the one already defined in (5.63),

$$F(\alpha_1(-m_1) \cdots \alpha_l(-m_l) e^k) = \bar{f}(\alpha_1)(-m_1) \cdots \bar{f}(\alpha_l)(-r_l) f(e^k), \tag{5.107}$$

and the action on the state (5.84) in $(V_L)_{\text{tw}}$ is defined as

$$F(c_1(-r_1) \cdots c_l(-r_l)|s\rangle) = \bar{f}(c_1)(-r_1) \cdots \bar{f}(c_l)(-r_l) f_{\mathcal{X}(L)}(|s\rangle). \tag{5.108}$$

As a result, $\hat{C}(L)$ acts on the whole $V_L \oplus (V_L)_{\text{tw}}$.

The group $C(L)$ acting on the $\mathbb{Z}_2$-orbifold $\tilde{V}_L = (V_L)^0 \oplus (V_L)^0_{\text{tw}}$

Now, we can see that $\hat{\theta}_0$ defined in (5.90) is $(\theta_0, (-1)^{\frac{n}{8}} \mathrm{id}_{\mathcal{X}(L)})$ in $\hat{C}(L)$. Therefore, the group $C(L) := \hat{C}(L)/\langle(\theta_0, (-1)^{\frac{n}{8}} \mathrm{id}_{\mathcal{X}(L)})\rangle$ acts on the $\hat{\theta}_0$-invariant sector $\tilde{V}_L = (V_L)^0 \oplus (V_L)^0_{\text{tw}}$. From (5.106), we have

$$1 \to Q \to C(L) \to O(L)/\{\pm\mathrm{id}_L\} \to 1. \tag{5.109}$$

When n is an even multiple of 8, then $C(L)$ coincides with $C_{\mathcal{X}(L)}$. When n is an odd multiple of 8, for example when L is the Leech lattice Λ_{24}, then $C(L)$ fits into the commutative diagram (5.105) as follows.

$$
\begin{array}{ccccc}
 & & \hat{C}(L) = 2^{1+n}.O(L) & & \\
 & \swarrow{\scriptstyle /\langle(\theta_0, \mathrm{id}_{\mathcal{X}(L)})\rangle} & \downarrow{\scriptstyle /\langle(\theta_0, -\mathrm{id}_{\mathcal{X}(L)})\rangle} & \searrow{\scriptstyle /\langle(\mathrm{id}_{\hat{L}}, -\mathrm{id}_{\mathcal{X}(L)})\rangle} & \\
C_{\mathcal{X}(L)} = 2^{1+n}.(O(L)/\{\pm\mathrm{id}_L\}) & & C(L) = 2^{1+n}.(O(L)/\{\pm\mathrm{id}_L\}) & & O(\hat{L}) = 2^n.O(L) \\
 & \searrow{\scriptstyle f \mapsto f \bullet f^{-1}} & \downarrow & \swarrow{\scriptstyle \varphi} & \\
 & & O(\hat{L})/\langle\theta_0\rangle = 2^n.(O(L)/\{\pm\mathrm{id}_L\}) & &
\end{array}
\tag{5.110}
$$

In particular, when L is the Leech lattice Λ_{24}, then $C(\Lambda_{24}) = 2^{1+24}.\mathrm{Co}_1$, which we mentioned in Section 2.4.

5.3.3 Triality and Shift $\mathbb{Z}_2$ Symmetry

The triality operator σ

From a given even self-dual lattice L, we have seen that we can construct two VOAs; one is the lattice VOA V_L, and the other is the orbifold $\tilde{V}_L$ of it by the reflection $\mathbb{Z}_2$ symmetry. We also explained in Section 2.3.1 that from a double-even self-dual binary code $\mathcal{C}$, we can construct two even self-dual lattices; one is $\Lambda(\mathcal{C})$ by Construction A, and the other is $\tilde{\Lambda}(\mathcal{C})$ by the twisted construction. As a result, from a double-even self-dual binary code $\mathcal{C}$, we can consider four VOAs, that is, $V_{\Lambda(\mathcal{C})}$, $V_{\tilde{\Lambda}(\mathcal{C})}$, $\tilde{V}_{\Lambda(\mathcal{C})}$, and $\tilde{V}_{\tilde{\Lambda}(\mathcal{C})}$. It was pointed out in [DGM90a, DGM94] that there are isomorphisms σ among these VOAs, and this will provide one description of the proof of $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$ as $\mathbb{M} = \langle C(\Lambda_{24}), \sigma\rangle$.

Recall that $\Lambda(\mathcal{C}) = \Lambda_0(\mathcal{C}) \sqcup \Lambda_1(\mathcal{C})$ and $\tilde{\Lambda}(\mathcal{C}) = \Lambda_0(\mathcal{C}) \sqcup \Lambda_3(\mathcal{C})$ in the notation[27] of (2.34)–(2.37). If we write the sector consisting of the states in the form of (5.29) with $k \in \Lambda_i(\mathcal{C})$ as $V_{\Lambda_i(\mathcal{C})}$, then

$$V_{\Lambda(\mathcal{C})} = V_{\Lambda_0(\mathcal{C})} \oplus V_{\Lambda_1(\mathcal{C})}, \tag{5.111}$$

$$V_{\tilde{\Lambda}(\mathcal{C})} = V_{\Lambda_0(\mathcal{C})} \oplus V_{\Lambda_3(\mathcal{C})}. \tag{5.112}$$

[27] The correspondence of the notations between [DGM94] and [FLM88, §12] is, in a situation where n is an odd multiple of 8,

[DGM94]	$\Lambda(\mathcal{C})$	$\Lambda_2(\mathcal{C}) \sqcup \Lambda_3(\mathcal{C})$	$\Lambda_0(\mathcal{C})$	$\Lambda_1(\mathcal{C})$	$\Lambda_2(\mathcal{C})$	$\Lambda_3(\mathcal{C})$	σ, σ_3	σ_1	Nothing
[FLM88]	L_0	L_1	Λ_0	Λ_2	Λ_3	Λ_1	σ	τ	σ_1

Our main text basically follows [DGM94], but we will use τ instead of σ_1 in Remark 5.9. We also use

these notes	$V_{\Lambda_i(\mathcal{C})}$	$V_{\mathcal{X}_i(\Lambda_0(\mathcal{C}))}$	$(i = 0, 1, 2, 3)$	$V_\bullet^0$	$V_\bullet^1$
[FLM88]	V_{Λ_j}	V'_{Λ_j}	$(j = 0, 2, 3, 1)$	$V_\bullet^+$	$V_\bullet^-$

In particular, since $\Lambda_0(\mathcal{C})$ is also a lattice, $V_{\Lambda_0(\mathcal{C})}$ forms a VOA.

We can also introduce similar decompositions to the twisted sectors $(V_{\Lambda(\mathcal{C})})_{\rm tw}$ and $(V_{\tilde{\Lambda}(\mathcal{C})})_{\rm tw}$ as follows. If we decompose the irreducible representation $\mathcal{X}(\Lambda(\mathcal{C}))$ of $\Gamma(\Lambda(\mathcal{C}))$ and that $\mathcal{X}(\tilde{\Lambda}(\mathcal{C}))$ of $\Gamma(\tilde{\Lambda}(\mathcal{C}))$ into irreducible representations of $\Gamma(\Lambda_0(\mathcal{C}))$, then we obtain $\mathcal{X}(\Lambda(\mathcal{C})) = \mathcal{X}_0(\Lambda_0(\mathcal{C})) \oplus \mathcal{X}_1(\Lambda_0(\mathcal{C}))$ and $\mathcal{X}(\tilde{\Lambda}(\mathcal{C})) = \mathcal{X}_0(\Lambda_0(\mathcal{C})) \oplus \mathcal{X}_3(\Lambda_0(\mathcal{C}))$, respectively. The dimensions of $\mathcal{X}(\Lambda(\mathcal{C}))$ and $\mathcal{X}(\tilde{\Lambda}(\mathcal{C}))$ are $2^{\frac{n}{2}}$, and those of $\mathcal{X}_i(\Lambda_0(\mathcal{C}))$ are $2^{\frac{n}{2}-1}$ (see Remark 5.8). If we write the sector consisting of the states in the form of (5.84) with $|s\rangle \in \mathcal{X}_i(\Lambda_0(\mathcal{C}))$ as $V_{\mathcal{X}_i(\Lambda_0(\mathcal{C}))}$, then

$$(V_{\Lambda(\mathcal{C})})_{\rm tw} = V_{\mathcal{X}_0(\Lambda_0(\mathcal{C}))} \oplus V_{\mathcal{X}_1(\Lambda_0(\mathcal{C}))}, \tag{5.113}$$
$$(V_{\tilde{\Lambda}(\mathcal{C})})_{\rm tw} = V_{\mathcal{X}_0(\Lambda_0(\mathcal{C}))} \oplus V_{\mathcal{X}_3(\Lambda_0(\mathcal{C}))}. \tag{5.114}$$

We further decompose these sectors into $\hat{\theta}_0$-even sectors, denoted by the superscript ^{0}, and $\hat{\theta}_0$-odd sectors, denoted by the superscript ^{1}. As a result, the four VOAs can be summarized into the following diagram.

$$\begin{array}{ccccccccc}
 & & & & V_{\Lambda(\mathcal{C})} & & \tilde{V}_{\tilde{\Lambda}(\mathcal{C})} & & \\
 & & & & \parallel & & \parallel & & \\
 & & & & (V_{\Lambda_0(\mathcal{C})})^0 & & (V_{\Lambda_0(\mathcal{C})})^0 & & \\
 & & & & \oplus & & \oplus & & \\
V_{\tilde{\Lambda}(\mathcal{C})} & = & (V_{\Lambda_0(\mathcal{C})})^0 & \oplus & (V_{\Lambda_0(\mathcal{C})})^1 & \oplus & (V_{\Lambda_3(\mathcal{C})})^0 & \oplus & (V_{\Lambda_3(\mathcal{C})})^1 \\
 & & & & \oplus & & \oplus & & \\
\tilde{V}_{\Lambda(\mathcal{C})} & = & (V_{\Lambda_0(\mathcal{C})})^0 & \oplus & (V_{\Lambda_1(\mathcal{C})})^0 & \oplus & (V_{\mathcal{X}_0(\Lambda_0(\mathcal{C}))})^0 & \oplus & (V_{\mathcal{X}_1(\Lambda_0(\mathcal{C}))})^0 \\
 & & & & \oplus & & \oplus & & \\
 & & & & (V_{\Lambda_1(\mathcal{C})})^1 & & (V_{\mathcal{X}_3(\Lambda_0(\mathcal{C}))})^0 & &
\end{array} \tag{5.115}$$

At the end of Section 5.3.1, we mentioned that $(V_L)^0$ is a sub-VOA of V_L, for an even lattice L. Since $\Lambda_0(\mathcal{C})$ is an even lattice, $(V_{\Lambda_0(\mathcal{C})})^0$ is a sub-VOA of $V_{\Lambda_0(\mathcal{C})}$. Moreover, every sector $(V_{\Lambda_i(\mathcal{C})})^p$ and $(V_{\mathcal{X}_i(\Lambda_0(\mathcal{C}))})^p$ ($i = 0, 1, 2, 3$ and $p = 0, 1$)[28] has the structure of irreducible representation of $(V_{\Lambda_0(\mathcal{C})})^0$ [DGM94, Prop. 7.1].

[DGM90a, DGM94] constructed a map σ acting on these sectors as

$$\begin{array}{ccccc}
 & \circlearrowright & \circlearrowright & & \\
 & (V_{\Lambda_0(\mathcal{C})})^0 & (V_{\Lambda_0(\mathcal{C})})^0 & & \\
\circlearrowright & & & & \\
(V_{\Lambda_0(\mathcal{C})})^0 & (V_{\Lambda_0(\mathcal{C})})^1 & (V_{\Lambda_3(\mathcal{C})})^0 & (V_{\Lambda_3(\mathcal{C})})^1 & \\
 & \downarrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & , \\
(V_{\Lambda_0(\mathcal{C})})^0 & (V_{\Lambda_1(\mathcal{C})})^0 & (V_{\mathcal{X}_0(\Lambda_0(\mathcal{C}))})^0 & (V_{\mathcal{X}_1(\Lambda_0(\mathcal{C}))})^0 & \\
\circlearrowleft & & & & \\
 & (V_{\Lambda_1(\mathcal{C})})^1 & (V_{\mathcal{X}_3(\Lambda_0(\mathcal{C}))})^0 & & \\
 & \circlearrowleft & \circlearrowleft & &
\end{array} \tag{5.116}$$

[28] See [DGM94] for the details of the sectors with $i = 2$.

and

$$\sigma : V_{\Lambda(\mathcal{C})} \to V_{\Lambda(\mathcal{C})}, \tag{5.117}$$

$$\sigma : V_{\tilde{\Lambda}(\mathcal{C})} \to \tilde{V}_{\Lambda(\mathcal{C})}, \quad \sigma : \tilde{V}_{\Lambda(\mathcal{C})} \to V_{\tilde{\Lambda}(\mathcal{C})}, \tag{5.118}$$

$$\sigma : \tilde{V}_{\tilde{\Lambda}(\mathcal{C})} \to \tilde{V}_{\tilde{\Lambda}(\mathcal{C})}, \tag{5.119}$$

are all isomorphisms of VOAs. This map σ is called the *triality operator*. This map mixes the untwisted sector and the twisted sector of $\tilde{V}_{\tilde{\Lambda}(\mathcal{C})}$.

Remark 5.9. While σ preserves the first and fourth rows of (5.116), there also exists a map τ (see footnote 27) which preserves the first and second rows of (5.116). These operators σ and τ generate a group $\langle \sigma, \tau \rangle$ isomorphic to the symmetric group S_3, hence the term triality operator. *(Remark ends.)*

For example, in the case where $\mathcal{C}$ is the binary Golay code G_{24}, we can see from Table 2.2 that $V_{\Lambda(\mathcal{C})}$ is the lattice VOA $V_{(A_1)^{24}}$ of the $(A_1)^{24}$ Niemeier lattice, $\tilde{V}_{\Lambda(\mathcal{C})} \cong V_{\tilde{\Lambda}(\mathcal{C})}$ is the Leech lattice VOA $V_{\Lambda_{24}}$, and $\tilde{V}_{\tilde{\Lambda}(\mathcal{C})}$ is the reflection $\mathbb{Z}_2$ orbifold $\tilde{V}_{\Lambda_{24}}$ of the Leech lattice VOA, that is, the monster VOA $V^\natural$.

The shift $\mathbb{Z}_2$ symmetry

We briefly review that, from the viewpoint of physics, the CFT $V_{\tilde{\Lambda}(\mathcal{C})}$ introduced above can be regarded as the orbifold of the CFT $V_{\Lambda(\mathcal{C})}$ by its shift $\mathbb{Z}_2$ symmetry.

If we consider an even self-dual lattice $\Lambda(\mathcal{C})$ constructed by Construction A from a doubly-even self-dual code $\mathcal{C}$, the resulting CFT has the *shift $\mathbb{Z}_2$ symmetry* with respect to the shift vector $\chi := \frac{1}{\sqrt{2}}\vec{1} \in \Lambda(\mathcal{C})$ as

$$X(z) \mapsto X(z) + \pi\chi, \tag{5.120}$$

$$V_k(z) \propto :e^{\sqrt{-1}k\cdot X(z)}: \;\mapsto e^{\sqrt{-1}\pi k\cdot\chi} V_k(z) = \begin{cases} V_k(z) & (k \in \Lambda_0(\mathcal{C})) \\ -V_k(z) & (k \in \Lambda_1(\mathcal{C})) \end{cases} . \tag{5.121}$$

In terms of $O(\hat{L}) \subset \mathrm{Aut}(V_L)$ in (5.61), if we define $\chi^* := \langle \chi, - \rangle \in \mathrm{Hom}(L, \mathbb{Z})$, the shift $\mathbb{Z}_2$ symmetry is the subgroup $\mathbb{Z}_2 \subset O(\hat{L})$ generated by $(\chi^* \bmod 2) \in \mathrm{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$.

Then we can also consider the states under the twisted boundary condition by this shift $\mathbb{Z}_2$ symmetry. The even and odd sectors of the untwisted and twisted sectors under the action of the shift $\mathbb{Z}_2$ symmetry can be described as (see for example [LS19, Appendix A], [KNO23b, §2.2])

shift $\mathbb{Z}_2$	untwisted	twisted
even	$V_{\Lambda_0(\mathcal{C})}$	$V_{\Lambda_3(\mathcal{C})}$
odd	$V_{\Lambda_1(\mathcal{C})}$	$V_{\Lambda_2(\mathcal{C})}$

. (5.122)

Recall that $V_{\Lambda_i(\mathcal{C})}$ denotes the sector consisting of the states in the form of (5.29) with $k \in \Lambda_i(\mathcal{C})$.

As a result, the orbifold of the CFT $V_{\Lambda(\mathcal{C})}$ by its shift $\mathbb{Z}_2$ symmetry consists of the even sectors $V_{\Lambda_0(\mathcal{C})} \oplus V_{\Lambda_3(\mathcal{C})}$, which is precisely the CFT $V_{\tilde{\Lambda}(\mathcal{C})}$. So we can say that the isomorphism $\sigma : \tilde{V}_{\Lambda(\mathcal{C})} \cong$

$V_{\tilde{\Lambda}(\mathcal{C})}$ in (5.118) is the isomorphism between the reflection $\mathbb{Z}_2$ orbifold and the shift $\mathbb{Z}_2$ orbifold of the lattice CFT $V_{\Lambda(\mathcal{C})}$ constructed from a doubly-even self-dual code $\mathcal{C}$.

In fact, the triality operator σ is constructed in [DGM94, §7] as follows. In the CFT $V_{\Lambda(\mathcal{C})}$, for each axis of the lattice $\Lambda(\mathcal{C})$, there exists an $\widehat{\mathfrak{su}}(2)_1$ current

$$J_a(z_1)J_b(z_2) \sim \frac{1}{2}\delta_{ab}\frac{1}{(z_1-z_2)^2} + \sqrt{-1}\epsilon_{abc}\frac{J_c(z_2)}{z_1-z_2}, \tag{5.123}$$

as

$$J_1(z) = \frac{\sqrt{-1}}{\sqrt{2}}\partial X^i(z) \qquad \in (V_{\Lambda_0(\mathcal{C})})^1, \tag{5.124}$$

$$J_2(z) = -\frac{1}{2}(V_{\sqrt{2}\mathbb{e}_i}(z) + V_{-\sqrt{2}\mathbb{e}_i}(z)) \quad \in (V_{\Lambda_1(\mathcal{C})})^0, \tag{5.125}$$

$$J_3(z) = \frac{\sqrt{-1}}{2}(V_{\sqrt{2}\mathbb{e}_i}(z) - V_{-\sqrt{2}\mathbb{e}_i}(z)) \in (V_{\Lambda_1(\mathcal{C})})^1. \tag{5.126}$$

The operator σ is constructed so that it implements the $\mathrm{SU}(2)$ rotation $J_1 \leftrightarrow J_2$ and $J_3 \mapsto -J_3$. As a result, the reflection $\mathbb{Z}_2$ symmetry $J_1 \mapsto -J_1$, $J_2 \mapsto J_2$, $J_3 \mapsto -J_3$ is essentially equivalent to the shift $\mathbb{Z}_2$ symmetry $J_1 \mapsto J_1$, $J_2 \mapsto -J_2$, $J_3 \mapsto -J_3$.

We refer the reader to [KNO23b, KY24] for the shift $\mathbb{Z}_2$ symmetries of more general lattice CFTs, and their orbifolds (and fermionizations).

5.3.4 $\mathbb{Z}_2$-orbifold and Fermionization

In the modern understanding of fermionization [Tac18, KTT19] (see also [HNT20, BSZ24]), we can uniformly treat $\mathbb{Z}_2$-orbifold and fermionization. Let us briefly review it.

In general, suppose a theory $\mathcal{T}$ has a non-anomalous $\mathbb{Z}_2$ symmetry, and the even and odd sectors of the untwisted and twisted sectors are

$\mathbb{Z}_2$	untwisted	twisted
even	S	U
odd	T	V

. (5.127)

By orbifolding $\mathcal{T}$ by the $\mathbb{Z}_2$ symmetry, a new $\mathbb{Z}_2$ symmetry emerges,[29] and we obtain the following orbifold theory:

(orbifold)	untwisted	twisted
even	S	T
odd	U	V

. (5.128)

[29] If we write the partition function of the theory $\mathcal{T}$ on a torus twisted by the original $\mathbb{Z}_2$ symmetry in spatial and temporal directions as $Z^{(\mathcal{T})}{}^{a_t}_{a_s}$ ($a_s, a_t \in \{0,1\}$; see also Section F.2.1), then the new $\mathbb{Z}_2$ symmetry of the orbifold theory $\tilde{\mathcal{T}}$ is $Z^{(\tilde{\mathcal{T}})}{}^{b_t}_{b_s} = \frac{1}{2}\sum_{a_s,a_t\in\{0,1\}}(-1)^{a_s b_t - a_t b_s} Z^{(\mathcal{T})}{}^{a_t}_{a_s}$. More generally, if the original theory $\mathcal{T}[A]$ is on a surface Σ and coupled to the $\mathbb{Z}_2$ gauge field $A \in H^1(M;\mathbb{Z}_2)$, then the orbifold theory is $\tilde{\mathcal{T}}[B] := \frac{1}{\sqrt{|H^1(\Sigma;\mathbb{Z}_2)|}}\sum_{A\in H^1(\Sigma;\mathbb{Z}_2)}(-1)^{\int_\Sigma A\smile B}\mathcal{T}[A]$. See for example [BT17, §2], [LOZ23, §2.5], [BSZ24, §2] for more details.

By fermionizing $\mathcal{T}$ by the $\mathbb{Z}_2$ symmetry, on the other hand, we obtain a fermionic CFT. There are two ways of fermionization:

$$\begin{array}{c|cc} \text{(fermionic)} & \text{NS} & \text{R} \\ \hline (-1)^F=+1 & S & U \\ (-1)^F=-1 & V & T \end{array} \quad , \quad \begin{array}{c|cc} \text{(fermionic}') & \text{NS} & \text{R} \\ \hline (-1)^F=+1 & S & T \\ (-1)^F=-1 & V & U \end{array} \quad . \tag{5.129}$$

Here, NS and R denote the Neveu-Schwarz sector and the Ramond sector, respectively. In terms of a fermionic field $\psi(z)$, the NS sector satisfies the periodic boundary condition $\psi(e^{2\pi\sqrt{-1}}z) = \psi(z)$, and the R sector satisfies the anti-periodic boundary condition[30] $\psi(e^{2\pi\sqrt{-1}}z) = -\psi(z)$.

Remark 5.10. It is tempting to regard the NS/R sectors as the untwisted/twisted sectors by some $\mathbb{Z}_2$ symmetry. In fact, in mathematics, the NS sector and the R sector are usually formulated as a VOSA and its canonically-twisted module respectively (see Section 6.1.1). However, from the perspective of physics, it is natural to distinguish the NS/R sectors and the untwisted/twisted sectors, as the former is specified by the spin structure, but the latter is specified by the $\mathbb{Z}_2$ gauge field of the theory. See also footnote 58. *(Remark ends.)*

Remark 5.11. As remarked in Section 5.2.1, if we used an even self-dual lattice of signature (r, s), then we obtain a bosonic lattice CFT of central charge $(c, \tilde{c}) = (r, s)$, which is modular invariant up to the phases (E.1, E.2) from the gravitational anomaly $\nu = 2(\tilde{c} - c)$. Recall that the signature (r, s) of an even self-dual lattice satisfies $r - s \equiv 0 \bmod 8$ [Ser73, Ch. V]. More generally, it is known that the gravitational anomaly of a bosonic CFT with central charge $(c, \tilde{c})$ satisfies $2(\tilde{c} - c) \equiv 0 \bmod 16$. Therefore, if we are given a fermionic CFT with central charge satisfying $2(\tilde{c} - c) \equiv 0 \bmod 16$, then we can bosonize it as a reverse operation of the fermionization [BSZ24]. *(Remark ends.)*

Lattice CFTs constructed from more general codes over finite fields, and their orbifolds and fermionizations, are investigated in [GJF18, Yah22, KY23a, KNO23b, KY24]. Recently, the construction of CFTs from quantum codes through Lorentzian lattices was established in [DS20a, DS20b], and has been developed in many directions; see for example [DS20c, DS21, BDR21, ACD22,DK22], [HKM22], [Fur22,Fur23], [BR23], [KNO22,AKN+23,KNO23a,KNO23b,AKN24]. We also note that [HM20] established a relation between a certain quantum code and a particular K3 CFT studied in [GTVW13] in the context of Mathieu moonshine.

[30] If we move to the cylinder coordinate $\sigma = \sigma^1 + \sqrt{-1}\sigma^2$ defined as $z = e^{-\sqrt{-1}\sigma}$, since $\psi_{\mathrm{cyl}}(\sigma) = \psi(z)(\frac{\partial z}{\partial \sigma})^{\frac{1}{2}}$, the NS sector is anti-periodic $\psi_{\mathrm{cyl}}(\sigma + 2\pi) = -\psi_{\mathrm{cyl}}(\sigma)$, and the R sector is periodic $\psi_{\mathrm{cyl}}(\sigma + 2\pi) = \psi_{\mathrm{cyl}}(\sigma)$.

6 Conway Moonshine and Clifford Module VOSA

The Conway moonshine was first established in [Dun07] as an $\mathcal{N} = 1$ VOSA $V^{f\natural}$ such that its automorphism group preserving the $\mathcal{N} = 1$ structure is isomorphic to the sporadic Conway group Co_1. Compared to the monstrous moonshine, which was led by the existence of the modular function, the Conway moonshine usually emphasizes the existence of such a VO(S)A with the Conway group symmetry itself. The Conway moonshine module $V^{f\natural}$ is a kind of $\mathbb{Z}_2$-orbifold of the VOSA describing 24 real free fermions. So we first review the construction of the VOSA of free fermions as a Clifford algebra module in Section 6.1, and then the construction of $V^{\natural}$ and the Conway group action on it in Section 6.2. We finally review the statement of the Conway moonshine in Section 6.3.

6.1 Clifford Module VOSA

In this Section 6.1, we review the construction of the VOSA of free fermions as a Clifford algebra module. In Section 6.1.1, we will introduce the Clifford algebras $\mathrm{Cliff}(\hat{\mathfrak{a}})$, $\mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}})$ and their modules $A(\mathfrak{a})$, $A(\mathfrak{a})_{\mathrm{tw}}$, which describe the NS and R sectors of the fermions, respectively. In Section 6.1.2, we will review the spin group action on the modules.

6.1.1 Clifford Module Construction

The *tensor algebra* of a $\mathbb{K}$-vector space V is

$$T(V) := \bigoplus_{d=0}^{\infty} V^{\otimes d}, \tag{6.1}$$

where $V^{\otimes 0} = \mathbb{K}$.

Let V be a $\mathbb{K}$-vector space with a quadratic form q (see Remark A.3 for the definition of a quadratic form). Its *Clifford algebra* $\mathrm{Cliff}(V, q)$ is the quotient of its tensor algebra by the relation[31] $u \otimes u \sim -q(u)$ for $u \in V$.

$$\mathrm{Cliff}(V, q) := T(V)/u \otimes u \sim -q(u). \tag{6.2}$$

If the characteristic of the field $\mathbb{K}$ is not 2, then by using the the associated form $\langle u, v\rangle := \frac{1}{2}(q(u+v) - q(u) - q(v))$ in the sense of (A.46), we have $q(u) = \langle u, u\rangle$, and the relation $u \otimes u = -q(u)$ is equivalent to

$$u \otimes v + v \otimes u = -2\langle u, v\rangle \tag{6.3}$$

for $u, v \in V$.

If the quadratic form q is 0, then the Clifford algebra is the *exterior algebra* $\bigwedge V$ of V, and the multiplication $\otimes$ is often denoted by $\wedge$. For example, $u \wedge v = -v \wedge u$ for $u, v \in V$.

[31] Another convention $u \otimes u \sim q(u)$ is also commonly used for the definition of $\mathrm{Cliff}(V, q)$.

Construct the VOSA

Let $\mathfrak{a}$ be an n-dimensional $\mathbb{C}$-vector space with a non-degenerate symmetric bilinear form $\langle -, - \rangle$. Define

$$\hat{\mathfrak{a}} := \bigoplus_{r \in \mathbb{Z}+\frac{1}{2}} \mathfrak{a}(r) \tag{6.4}$$

as the direct sum of copies of $\mathfrak{a}$ labeled by $r \in \mathbb{Z} + \frac{1}{2}$, and the symmetric bilinear form on $\hat{\mathfrak{a}}$ as the $\mathbb{C}$-bilinear extension of

$$\langle u(r), v(s) \rangle = \langle u, v \rangle \delta_{r+s,0} \quad (u, v \in \mathfrak{a},\ r, s \in \mathbb{Z} + \frac{1}{2}). \tag{6.5}$$

Here, $u(r)$ ($u \in \mathfrak{a}$) denotes an element in $\mathfrak{a}(r)$.

We define the Clifford algebra $\mathrm{Cliff}(\hat{\mathfrak{a}})$ of $\hat{\mathfrak{a}}$ as the one with respect to the quadratic form $q(\hat{u}) = \langle \hat{u}, \hat{u} \rangle$ ($\hat{u} \in \hat{\mathfrak{a}}$). That is,

$$\mathrm{Cliff}(\hat{\mathfrak{a}}) := T(\hat{\mathfrak{a}}) / \hat{u} \otimes \hat{u} \sim -\langle \hat{u}, \hat{u} \rangle. \tag{6.6}$$

We will omit $\otimes$ below. It is easy to see that the relation $\hat{u}\hat{u} = -\langle \hat{u}, \hat{u} \rangle$ is equivalent to

$$\{u(r), v(s)\} := u(r)v(s) + v(s)u(r) = -2\langle u(r), v(s) \rangle. \tag{6.7}$$

We introduce a polarization[32] $\hat{\mathfrak{a}} = \hat{\mathfrak{a}}^- \oplus \hat{\mathfrak{a}}^+$ as

$$\hat{\mathfrak{a}}^- := \bigoplus_{r \in (\mathbb{Z}+\frac{1}{2})_{<0}} \mathfrak{a}(r), \quad \hat{\mathfrak{a}}^+ := \bigoplus_{r \in (\mathbb{Z}+\frac{1}{2})_{>0}} \mathfrak{a}(r). \tag{6.8}$$

$\mathrm{Cliff}(\hat{\mathfrak{a}}^-) = \bigwedge \hat{\mathfrak{a}}^-$ and $\mathrm{Cliff}(\hat{\mathfrak{a}}^+) = \bigwedge \hat{\mathfrak{a}}^+$ are also defined in the same way as $\mathrm{Cliff}(\hat{\mathfrak{a}})$, and they are subalgebras of $\mathrm{Cliff}(\hat{\mathfrak{a}})$.

Take the $\mathrm{Cliff}(\hat{\mathfrak{a}}^+)$-module $\mathbb{C}$, whose basis is denoted by $|0\rangle$, on which $u(r) \in \mathrm{Cliff}(\hat{\mathfrak{a}}^+)$ acts as $u(r)|0\rangle = 0$ ($u \in V, r > 0$), and $\mathbb{C} \subset \mathrm{Cliff}(\hat{\mathfrak{a}}^+)$ acts as the usual multiplication. We define the $\mathrm{Cliff}(\hat{\mathfrak{a}})$-module $A(\mathfrak{a})$ as the induced module from it:

$$A(\mathfrak{a}) := \mathrm{Cliff}(\hat{\mathfrak{a}}) \otimes_{\mathrm{Cliff}(\hat{\mathfrak{a}}^+)} \mathbb{C}|0\rangle. \tag{6.9}$$

That is, $\hat{\mathfrak{a}}^+$ annihilates $|0\rangle$, and $A(\mathfrak{a})$ is isomorphic to $\mathrm{Cliff}(\hat{\mathfrak{a}}^-)|0\rangle$ as a $\mathrm{Cliff}(\hat{\mathfrak{a}}^-)$-module.

We can equip $A(\mathfrak{a})$ with a VOSA structure of central charge $\frac{n}{2}$. Let $\psi_1, \ldots, \psi_n$ be an orthonormal[33] basis of $\mathfrak{a}$ with $\langle \psi_i, \psi_i \rangle = +1$. The Virasoro element is given as

$$\omega = -\frac{1}{4} \sum_{i=1}^{n} \psi_i(-\frac{3}{2}) \psi_i(-\frac{1}{2}) |0\rangle. \tag{6.10}$$

[32] An *isotropic* vector of a vector space V with a quadratic form q is a vector $v \in V$ (sometimes required to be non-zero) such that $q(v) = 0$. A *polarization* of V is a decomposition $V = V^+ \oplus V^-$ into maximal isotropic subspaces with respect to q. Note that the associated form of the quadratic form (A.45) is also called polarization, although we do not use this terminology in these notes to avoid confusion.

[33] Since we are considering the symmetric bilinear form $\langle -, - \rangle$, not a Hermitian form, on the $\mathbb{C}$-vector space, there is no concept of the signature of the form, and we can always take an orthonormal basis with squared norm $+1$.

The weight of the state $u_1(-r_1)\cdots u_l(-r_l)|0\rangle$ ($u_i \in \mathfrak{a}$, $r_i \in (\mathbb{Z}+\frac{1}{2})_{>0}$) is

$$\mathrm{wt}(u_1(-r_1)\cdots u_l(-r_l)|0\rangle) = r_1 + \cdots + r_l. \tag{6.11}$$

The map $-\mathrm{id}_{\hat{\mathfrak{a}}} : \hat{\mathfrak{a}} \to \hat{\mathfrak{a}}$ induces the map $\theta : \mathrm{Cliff}(\hat{\mathfrak{a}}) \to \mathrm{Cliff}(\hat{\mathfrak{a}})$ called the *parity involution*. We write its eigenspace decomposition as $\mathrm{Cliff}(\hat{\mathfrak{a}}) = \mathrm{Cliff}(\hat{\mathfrak{a}})^0 \oplus \mathrm{Cliff}(\hat{\mathfrak{a}})^1$ with eigenvalues 1 and -1 respectively. We introduce the parity decomposition $A(\mathfrak{a}) = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1$ by

$$A(\mathfrak{a})^0 := \mathrm{Cliff}(\hat{\mathfrak{a}})^0|0\rangle = (\bigwedge^{\text{even}} \hat{\mathfrak{a}}^-)|0\rangle, \tag{6.12}$$

$$A(\mathfrak{a})^1 := \mathrm{Cliff}(\hat{\mathfrak{a}})^1|0\rangle = (\bigwedge^{\text{odd}} \hat{\mathfrak{a}}^-)|0\rangle. \tag{6.13}$$

This introduces the $\mathbb{Z}_2$-grading of the VOSA on $A(\mathfrak{a})$. Any states in $A(\mathfrak{a})^0$ have integer weights, and states in $A(\mathfrak{a})^1$ have half-integer weights. In particular, the subalgebra $A(\mathfrak{a})^0$ of $A(\mathfrak{a})$ is a VOA.

In physics notation, $\psi_i(r)$ is often denoted by ψ^i_r, and the elements of $\hat{\mathfrak{a}}^-$ and $\hat{\mathfrak{a}}^+$ in the form of $u(r)$ are called the creation operators and the annihilation operators, respectively. $A(\mathfrak{a})$ is the NS sector of the n real free chiral fermion theory.

Construct the R sector

In general, for a vertex superalgebra $V = V^{\bar{0}} \oplus V^{\bar{1}}$, its *canonically-twisted module* M is a $\mathbb{Z}_2$-twisted module M of V with $\mathbb{Z}_2$-grading $M^{\bar{0}} \oplus M^{\bar{1}}$ such that the twisted vertex operators (4.18)

$$Y^M(v,z) = \sum_{n\in\frac{1}{2}\mathbb{Z}} v^M_{(n)} z^{-n-1} \tag{6.14}$$

satisfy, for any $v \in V^{\bar{p}}$, $v^M_{(n)}$ maps $M^{\bar{q}}$ to $M^{\overline{p+q}}$, and $v^M_{(n)} = 0$ if $n \notin \mathbb{Z}+\frac{p}{2}$. Any vertex superalgebra has the parity involution $\theta := \mathrm{id}_{V^{\bar{0}}} \oplus (-\mathrm{id}_{V^{\bar{1}}})$, and in light of this, a canonically-twisted module is a θ-twisted supermodule.

We can construct a canonically-twisted module $A(\mathfrak{a})_{\mathrm{tw}}$ of $A(\mathfrak{a})$ as follows. It is the R sector of the n real free chiral fermion theory in the language of physics.

We assume $\underline{n = \dim\mathfrak{a} \text{ is even}}$ below. Define

$$\hat{\mathfrak{a}}_{\mathrm{tw}} := \bigoplus_{r\in\mathbb{Z}} \mathfrak{a}(r) \tag{6.15}$$

and the bilinear form on $\hat{\mathfrak{a}}_{\mathrm{tw}}$ as

$$\langle u(r), v(s)\rangle = \langle u,v\rangle \delta_{r+s,0} \quad (u,v\in\mathfrak{a},\ r,s\in\mathbb{Z}). \tag{6.16}$$

We choose a polarization of $\mathfrak{a}$ as $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$. They are usually chosen so that

$$\mathfrak{a}^- = \mathrm{Span}_{\mathbb{C}}\{\Psi_1,\ldots,\Psi_{\frac{n}{2}}\}, \quad \mathfrak{a}^+ = \mathrm{Span}_{\mathbb{C}}\{\overline{\Psi}_1,\ldots,\overline{\Psi}_{\frac{n}{2}}\}, \tag{6.17}$$

where

$$\Psi_i := \frac{1}{\sqrt{2}}(\psi_{2i-1} + \sqrt{-1}\psi_{2i}), \quad \overline{\Psi}_i := \frac{1}{\sqrt{2}}(\psi_{2i-1} - \sqrt{-1}\psi_{2i}), \tag{6.18}$$

for an orthonormal basis $\psi_1, \ldots, \psi_n$ of $\mathfrak{a}$. Then we introduce a polarization $\hat{\mathfrak{a}}_{\mathrm{tw}} = \hat{\mathfrak{a}}_{\mathrm{tw}}^- \oplus \hat{\mathfrak{a}}_{\mathrm{tw}}^+$ as

$$\hat{\mathfrak{a}}_{\mathrm{tw}}^- := \mathfrak{a}^-(0) \oplus \bigoplus_{r \in \mathbb{Z}_{<0}} \mathfrak{a}(r), \quad \hat{\mathfrak{a}}_{\mathrm{tw}}^+ := \mathfrak{a}^+(0) \oplus \bigoplus_{r \in \mathbb{Z}_{>0}} \mathfrak{a}(r). \tag{6.19}$$

Again, $\mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}}^\pm) = \bigwedge \hat{\mathfrak{a}}_{\mathrm{tw}}^\pm$ are subalgebras of $\mathrm{Cliff}(\hat{\mathfrak{a}})$.

Take the $\mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}}^+)$-module $\mathbb{C}$, whose basis is denoted by $|0\rangle_{\mathrm{R}}$, on which $u(r) \in \mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}}^+)$ acts as $u(r)|0\rangle_{\mathrm{R}} = 0$, and $\mathbb{C} \subset \mathrm{Cliff}(\hat{\mathfrak{a}}^+)$ acts as the usual multiplication. We define the $\mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}})$-module $A(\mathfrak{a})_{\mathrm{tw}}$ as

$$A(\mathfrak{a})_{\mathrm{tw}} := \mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}}) \otimes_{\mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}}^+)} \mathbb{C}|0\rangle_{\mathrm{R}}. \tag{6.20}$$

That is, $\hat{\mathfrak{a}}_{\mathrm{tw}}^+$ annihilates $|0\rangle_{\mathrm{R}}$.

We can equip $A(\mathfrak{a})_{\mathrm{tw}}$ with a canonically-twisted $A(\mathfrak{a})$-module structure. The weight of the state $u_1(-r_1) \cdots u_l(-r_l) u_1^-(0) \cdots u_{l'}(0)|0\rangle_{\mathrm{R}}$ ($u_i \in \mathfrak{a}$, $r_i \in \mathbb{Z}_{>0}$, $u_i^- \in \mathfrak{a}^-$) is

$$\mathrm{wt}(u_1(-r_1) \cdots u_l(-r_l) u_1^-(0) \cdots u_{l'}(0)|0\rangle_{\mathrm{R}}) = r_1 + \cdots + r_l + \frac{n}{16}. \tag{6.21}$$

The $\mathbb{Z}_2$-grading on $A(\mathfrak{a})_{\mathrm{tw}}$ is introduced as follows: we assign 0 or 1 to the parity of $|0\rangle_{\mathrm{R}}$, and then $(\bigwedge^{\mathrm{even}} \hat{\mathfrak{a}}_{\mathrm{tw}}^-)|0\rangle_{\mathrm{R}}$ has the same parity as $|0\rangle_{\mathrm{R}}$, whereas $(\bigwedge^{\mathrm{odd}} \hat{\mathfrak{a}}_{\mathrm{tw}}^-)|0\rangle_{\mathrm{R}}$ has the opposite parity. The parity of $|0\rangle_{\mathrm{R}}$ is basically arbitrary. For example, [FFR91] always defines the parity of $|0\rangle_{\mathrm{R}}$ as 0. In these notes, following [DMC14], we specify it as $\frac{n}{4} \bmod 2$, when n is a multiple of 4, which will be justified in Section 6.1.2 as the eigenvalue $(-1)^{\frac{n}{4}}$ of $|0\rangle_{\mathrm{tw}}$ with respect to the action by the lift $\psi_1 \cdots \psi_n \in \mathrm{Spin}(\mathfrak{a})$ of $-\mathrm{id}_{\mathfrak{a}} \in \mathrm{SO}(\mathfrak{a})$. We write the subspaces of $A(\mathfrak{a})_{\mathrm{tw}}$ with parity 0 and 1 as $A(\mathfrak{a})_{\mathrm{tw}}^0$ and $A(\mathfrak{a})_{\mathrm{tw}}^1$, respectively.

6.1.2 Spin Group Action

Spin group $\mathrm{Spin}(\mathfrak{a})$

The details of the definition and properties of the spin group are reviewed in Appendix D. Here, we only present the main points.

Recall that $\mathfrak{a}$ is an n-dimensional $\mathbb{C}$-vector space with symmetric bilinear form $\langle -, - \rangle$. Similarly to $\mathrm{Cliff}(\hat{\mathfrak{a}})$, we introduce the parity decomposition $\mathrm{Cliff}(\mathfrak{a}) = \mathrm{Cliff}(\mathfrak{a})^0 \oplus \mathrm{Cliff}(\mathfrak{a})^1$ induced by $-\mathrm{id}_{\mathfrak{a}}$. The Clifford algebra $\mathrm{Cliff}(\mathfrak{a})$ contains the *spin group* $\mathrm{Spin}(\mathfrak{a})$ defined by

$$\mathrm{Spin}(\mathfrak{a}) := \{u_1 \cdots u_{2l} \mid u_i \in \mathfrak{a}, \langle u_i, u_i \rangle = 1\} \tag{6.22}$$

$$= \{x \in (\mathrm{Cliff}(\mathfrak{a})^0)^\times \mid x\mathfrak{a}x^{-1} \subset \mathfrak{a}, x^T x = 1\}. \tag{6.23}$$

Here, $(\mathrm{Cliff}(\mathfrak{a})^0)^\times$ is the set of invertible elements of $\mathrm{Cliff}(\mathfrak{a})^0$, and x^T is defined as $x^T = u_l \cdots u_1$ if $x = u_1 \cdots u_l$ ($u_i \in \mathfrak{a}$). If $x \in \mathbb{C}$, then $x^T = x$. (This is the definition of the spin group in the sense of (D.25, D.27).)

An important property of $\mathrm{Spin}(\mathfrak{a})$ is the following fact.

Proposition 6.1. $\mathrm{Spin}(\mathfrak{a})$ *is a double cover of* $\mathrm{SO}(\mathfrak{a})$ *defined as*

$$\mathrm{SO}(\mathfrak{a}) := \{O \in \mathrm{SL}(\mathfrak{a}) \mid \langle Ou, Ov\rangle = \langle u, v\rangle \text{ for any } u, v \in \mathfrak{a}\}. \tag{6.24}$$

More precisely, $\mathrm{Spin}(\mathfrak{a})$ *is a central extension of* $\mathrm{SO}(\mathfrak{a})$ *by* $\mathbb{Z}_2$

$$1 \to \{\pm 1_{\mathrm{Spin}(\mathfrak{a})}\} \hookrightarrow \mathrm{Spin}(\mathfrak{a}) \xrightarrow{\varphi} \mathrm{SO}(\mathfrak{a}) \to 1, \tag{6.25}$$

where $\varphi : \mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$ *is the group homomorphism defined as*

$$\varphi : x \mapsto x \bullet x^{-1}. \tag{6.26}$$

In particular, if we take an orthonormal basis $\psi_1, \ldots, \psi_n$ *of* $\mathfrak{a}$*, then the lifts* $\pm\hat{O} \in \mathrm{Spin}(\mathfrak{a})$ *of* $O \in \mathrm{SO}(\mathfrak{a})$ *act on* $\sum_i c^i\psi_i \in \mathfrak{a}$ *as (D.34)*

$$(\pm\hat{O}) \left[\begin{pmatrix}\psi_1 \cdots \psi_n\end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix} \right] (\pm\hat{O})^{-1} = \begin{pmatrix}\psi_1 \cdots \psi_n\end{pmatrix} O \begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix}, \tag{6.27}$$

where O *is represented as a matrix with respect to the basis* $\{\psi_i\}_i$.

The $\mathrm{Spin}(\mathfrak{a})$**-action on** $A(\mathfrak{a})$ **and** $A(\mathfrak{a})_{\mathrm{tw}}$

The Clifford module VOSA $A(\mathfrak{a})$ and its twisted module $A(\mathfrak{a})_{\mathrm{tw}}$ naturally admit a $\mathrm{Spin}(\mathfrak{a})$-action as follows.

The action of $x \in \mathrm{Spin}(\mathfrak{a})$ on $u_1(r_1)\cdots u_l(r_l)|0\rangle \in A(\mathfrak{a})$ ($r_i \in \mathbb{Z} + \frac{1}{2}$) is defined as

$$u_1(r_1)\cdots u_k(r_k)|0\rangle \;\mapsto\; (xu_1x^{-1})(r_1)\cdots(xu_kx^{-1})(r_k)|0\rangle. \tag{6.28}$$

This action is well-defined because $\{(xu_ix^{-1})(r_i), (xu_jx^{-1})(r_j)\} = -2\langle xu_ix^{-1}, xu_jx^{-1}\rangle\delta_{r_i+r_j,0} = \{u_i(r_i), u_j(r_j)\}$. Since $-1_{\mathrm{Spin}(\mathfrak{a})} \in \mathrm{Spin}(\mathfrak{a})$ acts on $A(\mathfrak{a})$ trivially, this $\mathrm{Spin}(\mathfrak{a})$-action reduces to $\mathrm{Spin}(\mathfrak{a})/\langle -1_{\mathrm{Spin}(\mathfrak{a})}\rangle \cong \mathrm{SO}(\mathfrak{a})$-action.

To describe the action of $\mathrm{Spin}(\mathfrak{a})$ on $A(\mathfrak{a})_{\mathrm{tw}}$, we identify $\mathrm{Cliff}(\mathfrak{a})$ with $\mathrm{Cliff}(\mathfrak{a}(0)) \subset \mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}})$. Then $x \in \mathrm{Spin}(\mathfrak{a}) \subset \mathrm{Cliff}(\mathfrak{a})$ acts on $|0\rangle_{\mathrm{R}}$. So the action of $x \in \mathrm{Spin}(\mathfrak{a})$ on $u_1(r_1)\cdots u_l(r_l)|0\rangle_{\mathrm{R}} \in A(\mathfrak{a})_{\mathrm{tw}}$ ($r_i \in \mathbb{Z}$) is defined as

$$u_1(r_1)\cdots u_k(r_k)|0\rangle_{\mathrm{R}} \;\mapsto\; (xu_1x^{-1})(r_1)\cdots(xu_kx^{-1})(r_k)x|0\rangle_{\mathrm{R}}. \tag{6.29}$$

The action of $-1_{\mathrm{Spin}(\mathfrak{a})} \in \mathrm{Spin}(\mathfrak{a})$ on $A(\mathfrak{a})_{\mathrm{tw}}$ is nontrivial, and $A(\mathfrak{a})_{\mathrm{tw}}$ is only a projective representation of $\mathrm{SO}(\mathfrak{a})$.

There are two lifts of $-1_{\mathrm{SO}(\mathfrak{a})} \in \mathrm{SO}(\mathfrak{a})$. If we take an orthonormal basis $\psi_1, \ldots, \psi_n$ of $\mathfrak{a}$, then they can be explicitly written as $\pm\psi_1\cdots\psi_n \in \mathrm{Spin}(\mathfrak{a})$. Depending on the choice of the polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$, one acts as $|0\rangle_{\mathrm{R}} \mapsto \sqrt{-1}^{\frac{n}{2}}|0\rangle_{\mathrm{R}}$, and the other acts as $|0\rangle_{\mathrm{R}} \mapsto -\sqrt{-1}^{\frac{n}{2}}|0\rangle_{\mathrm{R}}$. The former one is denoted by $\mathfrak{z}$ and called the *lift of* $-1_{\mathrm{SO}(\mathfrak{a})}$ *associated with the polarization.* When $n = \dim\mathfrak{a}$ is a multiple of 4, the $\mathbb{Z}_2$-grading of the canonically-twisted module

$A(\mathfrak{a})_{\mathrm{tw}} = A(\mathfrak{a})^0_{\mathrm{tw}} \oplus A(\mathfrak{a})^1_{\mathrm{tw}}$ introduced at the end of the previous Section 6.1.1 coincides with the eigenspace decomposition with respective to $\mathfrak{z}$ with eigenvalue $+1$ and -1. The $\mathbb{Z}_2$-grading of the VOSA $A(\mathfrak{a}) = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1$ also coincides with the eigenspace decomposition with respect to $\mathfrak{z}$. The eigenvalues of each sector with respect to $\mathfrak{z}$, $-\mathfrak{z}$, and $-1_{\mathrm{Spin}(\mathfrak{a})}$ can be summarized as follows.

	$A(\mathfrak{a})^0$	$A(\mathfrak{a})^1$	$A(\mathfrak{a})^0_{\mathrm{tw}}$	$A(\mathfrak{a})^1_{\mathrm{tw}}$
eigenvalue w.r.t. $\mathfrak{z}$	1	-1	1	-1
eigenvalue w.r.t. $-\mathfrak{z}$	1	-1	-1	1
eigenvalue w.r.t. $-1_{\mathrm{Spin}(\mathfrak{a})}$	1	1	-1	-1

(6.30)

If we choose the polarization as in (6.17), then we can explicitly check

$$\mathfrak{z} = +\psi_1 \cdots \psi_n = \sqrt{-1}^{\frac{n}{2}} (\Psi_1 \overline{\Psi}_1 + 1) \cdots (\Psi_{\frac{n}{2}} \overline{\Psi}_{\frac{n}{2}} + 1). \tag{6.31}$$

Finally, we remark that these $\mathrm{Spin}(\mathfrak{a})$-actions on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\mathrm{tw}}$ are weight-preserving and parity-preserving.

6.2 Duncan's Supermoonshine Module

In this Section 6.2, we review the construction of the Conway moonshine module $V^{f\natural}$. It is constructed from the VOSA of free fermions in a way similar to the $\mathbb{Z}_2$-orbifold in Section 6.2.1, and we will see how the Conway group action is realized on it in Section 6.2.2.

6.2.1 Construction of Duncan's Module

We set $\mathfrak{a} = \Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{C}$ where Λ_{24} is the Leech lattice with symmetric bilinear form $\langle -, - \rangle$. In particular, $n = \dim \mathfrak{a} = 24$. The $A(\mathfrak{a})^0$-module structure on

$$V^{f\natural} := A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^0_{\mathrm{tw}} \tag{6.32}$$

extends uniquely to an VOSA structure, and the $A(\mathfrak{a})^0$-module structure on

$$V^{f\natural}_{\mathrm{tw}} := A(\mathfrak{a})^1 \oplus A(\mathfrak{a})^1_{\mathrm{tw}} \tag{6.33}$$

extends uniquely to a canonically-twisted $V^{f\natural}$-module structure. There seems no consensus on the name of this VOSA $V^{f\natural}$, or the whole theory $V^{f\natural} \oplus V^{f\natural}_{\mathrm{tw}}$. We call it the *Conway (super)moonshine module* following [TW17], or *Duncan's (supermoonshine) module* following [AKL22], because it was first constructed by Duncan in [Dun07], although it should be noted that it was also revisited with Mack-Crane in [DMC14].

The original theory $A(\mathfrak{a})$ and Duncan's module $V^{f\natural}$ are in the relation similar to the $\mathbb{Z}_2$-orbifold[34] of each other.

$$\begin{array}{c|cc} & \text{NS} & \text{R} \\ \mathbb{Z}_2 = \langle \mathfrak{z} \rangle & A(\mathfrak{a}) & A(\mathfrak{a})_{\mathrm{tw}} \\ \hline \text{even} & A^0(\mathfrak{a}) & A^0(\mathfrak{a})_{\mathrm{tw}} \\ \text{odd} & A^1(\mathfrak{a}) & A^1(\mathfrak{a})_{\mathrm{tw}} \end{array} \quad \underset{\text{`}\langle -1_{\mathrm{Spin}(\mathfrak{a})}\rangle\text{-orbifold'}}{\overset{\text{`}\langle \mathfrak{z} \rangle\text{-orbifold'}}{\rightleftarrows}} \quad \begin{array}{c|cc} & \text{NS} & \text{R} \\ \mathbb{Z}_2 = \langle -1_{\mathrm{Spin}(\mathfrak{a})} \rangle & V^{f\natural} & V^{f\natural}_{\mathrm{tw}} \\ \hline \text{even} & A^0(\mathfrak{a}) & A^1(\mathfrak{a}) \\ \text{odd} & A^0(\mathfrak{a})_{\mathrm{tw}} & A^1(\mathfrak{a})_{\mathrm{tw}} \end{array} \tag{6.34}$$

We may consider the similar construction with respect to $-\mathfrak{z}$ (6.30), and the resulting theory is denoted by $V^{s\natural}$.

$$\begin{array}{c|cc} & \text{NS} & \text{R} \\ \mathbb{Z}_2 = \langle -\mathfrak{z} \rangle & A(\mathfrak{a}) & A(\mathfrak{a})_{\mathrm{tw}} \\ \hline \text{even} & A^0(\mathfrak{a}) & A^1(\mathfrak{a})_{\mathrm{tw}} \\ \text{odd} & A^1(\mathfrak{a}) & A^0(\mathfrak{a})_{\mathrm{tw}} \end{array} \quad \underset{\text{`}\langle -1_{\mathrm{Spin}(\mathfrak{a})}\rangle\text{-orbifold'}}{\overset{\text{`}\langle -\mathfrak{z} \rangle\text{-orbifold'}}{\rightleftarrows}} \quad \begin{array}{c|cc} & \text{NS} & \text{R} \\ \mathbb{Z}_2 = \langle -1_{\mathrm{Spin}(\mathfrak{a})} \rangle & V^{s\natural} & V^{s\natural}_{\mathrm{tw}} \\ \hline \text{even} & A^0(\mathfrak{a}) & A^1(\mathfrak{a}) \\ \text{odd} & A^1(\mathfrak{a})_{\mathrm{tw}} & A^0(\mathfrak{a})_{\mathrm{tw}} \end{array} \tag{6.35}$$

The $A(\mathfrak{a})^0$-module structure on

$$V^{s\natural} := A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{\mathrm{tw}} \tag{6.36}$$

extends uniquely to an VOSA structure, and the $A(\mathfrak{a})^0$-module structure on

$$V^{s\natural}_{\mathrm{tw}} := A(\mathfrak{a})^1 \oplus A(\mathfrak{a})^0_{\mathrm{tw}} \tag{6.37}$$

extends uniquely to a canonically-twisted $V^{s\natural}$-module structure. This $V^{s\natural}$ is isomorphic to $V^{f\natural}$ as a VOSA, and also called the Conway moonshine module.

Remark 6.2 (lattice description). Through the boson-fermion correspondence [Fre81], the VOA $A(\mathfrak{a})^0$ is isomorphic to the lattice VOA $V_{D_{12}}$ of the D_{12} lattice. Recall that

$$D_n := \{(k_1, \ldots, k_n) \in \mathbb{Z}^n \mid \sum_{i=1}^{n} k_i \equiv 0 \bmod 2\}. \tag{6.38}$$

It is known that the irreducible representations of a lattice VOA V_L for an even lattice L are in one-to-one correspondence with the elements of the quotient L^*/L of the dual lattice L^*, called the *discriminant group*[35] of the lattice L, and labeled as $\{V_{\lambda+L}\}_{\lambda+L\in L^*/L}$ [Don93].

The D_n lattice is $\sqrt{2}\times$ the lattice constructed by Construction A from the *even weight code* $\mathcal{E}_n := \{w \in \mathbb{F}_2^n \mid \mathrm{wt}(w) \equiv 0 \bmod 2\}$ [CS99, Ch. 3 §2.3], or $\sqrt{2}\times$ the lattice constructed by

[34] In the viewpoint of footnote 58, we assume that the spin structures on the tori of both the original and orbifold theories are the NS sectors (in the spatial direction), and take the $\mathbb{Z}_2$-orbifold by the $\mathbb{Z}_2$ gauge field. In this sense, $A(\mathfrak{a})_{\mathrm{tw}}$ is in fact the $\mathbb{Z}_2$-twisted sector of the NS sector $A(\mathfrak{a})$. The reason why we can also regard $A(\mathfrak{a})_{\mathrm{tw}}$ as the R sector is that the fermions are subject to the sum $\sigma + A$ of the spin structure σ and the $\mathbb{Z}_2$ gauge field A.

[35] If a lattice L is integral, we have $L \subset L^*$, and hence the discriminant group L^*/L is well-defined. Letting G denote the Gram matrix L, we have $|L^*/L| = \det G$.

Construction B from the trivial code 0_n of length n. Mimicking (2.34)–(2.37) for doubly-even self dual code $\mathcal{C}$, let us define for 0_n with $n \equiv 0 \bmod 4$,

$$\Lambda_0(0_n) := \mathbb{Z}^n_+ = D_n, \tag{6.39}$$

$$\Lambda_1(0_n) := \mathbb{Z}^n_- = (1, 0, \dots, 0) + D_n, \tag{6.40}$$

$$\Lambda_2(0_n) := \frac{1}{2}\vec{1} + \mathbb{Z}^n_{(-)^{\frac{n}{4}+1}} = \frac{1}{2}((-1)^{\frac{n}{4}+1}, 1, \dots, 1) + D_n, \tag{6.41}$$

$$\Lambda_3(0_n) := \frac{1}{2}\vec{1} + \mathbb{Z}^n_{(-)^{\frac{n}{4}}} = \frac{1}{2}((-1)^{\frac{n}{4}}, 1, \dots, 1) + D_n. \tag{6.42}$$

Now, we have $D_{12} = \Lambda_0(0_{12})$, and $D^*_{12} = \bigsqcup_{i=0}^{3} \Lambda_i(0_{12})$. $|D^*_{12}/D_{12}| = 4$ and the four elements of D^*_{12}/D_{12} are $\Lambda_i(0_{12})$ $(i = 0, 1, 2, 3)$. As $A(\mathfrak{a})^0$-modules,

$$A(\mathfrak{a})^0 \cong V_{\Lambda_0(0_{12})}, \quad A(\mathfrak{a})^1 \cong V_{\Lambda_2(0_{12})}, \quad A(\mathfrak{a})^0_{\mathrm{tw}} \cong V_{\Lambda_2(0_{12})}, \quad A(\mathfrak{a})^1_{\mathrm{tw}} \cong V_{\Lambda_3(0_{12})}. \tag{6.43}$$

As a result, $V^{f\natural} \cong V_{D^+_{12}}$ as VOSAs, where $D^+_{12} := D_{12} \sqcup (\frac{1}{2}\vec{1} + D_{12})$ is the unique self-dual positive-definite lattice of rank 12 without vectors of squared length 1. Since $D^+_{12} \cong D_{12} \sqcup (\frac{1}{2}(-1, 1, \dots, 1) + D_{12})$, we can see $V^{s\natural} \cong V_{D^+_{12}} \cong V^{f\natural}$. *(Remark ends.)*

6.2.2 Conway Group Action

The $\mathrm{SemiSpin}(24)$-action on $V^{f\natural}$

We focus on $\mathrm{SO}(24) = \mathrm{SO}(\Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{R}) \subset \mathrm{SO}(\mathfrak{a})$, and its double cover $\mathrm{Spin}(24) = \mathrm{Spin}(\Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{R}) \subset \mathrm{Spin}(\mathfrak{a})$.

In general, if n is a multiple of 4, then the center[36] of $\mathrm{Spin}(n)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle -1_{\mathrm{Spin}(n)} \rangle \times \langle \mathfrak{z} \rangle$. The quotient $\mathrm{Spin}(n)/\langle -1_{\mathrm{Spin}(n)} \rangle$ is isomorphic to $\mathrm{SO}(n)$, whereas the quotient $\mathrm{Spin}(n)/\langle \mathfrak{z} \rangle$ is another group called the *semi-spin group* $\mathrm{SemiSpin}(n)$, not isomorphic to $\mathrm{SO}(n)$ except for $n = 8$.

Recall that $\mathrm{Spin}(24)$ acts on the whole $A(\mathfrak{a}) \oplus A(\mathfrak{a})_{\mathrm{tw}}$. Since the action of $\mathfrak{z}$ on $V^{f\natural}$ is trivial, the $\mathrm{Spin}(24)$-action on $V^{f\natural}$ reduces to the action by the quotient $\mathrm{Spin}(24)/\langle -\mathfrak{z} \rangle \cong \mathrm{SemiSpin}(24)$.

$$\begin{array}{ccc} & \mathrm{Spin}(24) \curvearrowright A(\mathfrak{a}) \oplus A(\mathfrak{a})_{\mathrm{tw}} & \\ \swarrow{\scriptstyle /\langle \mathfrak{z} \rangle} & & \searrow{\scriptstyle /\langle -1_{\mathrm{Spin}(24)} \rangle} \\ \mathrm{SemiSpin}(24) \curvearrowright V^{f\natural} & & \mathrm{SO}(24) \curvearrowright A(\mathfrak{a}) \end{array} \tag{6.45}$$

[36] The center of $\mathrm{Spin}(n)$ is [Bin13, Ch. 8]

$$Z(\mathrm{Spin}(n)) = \begin{cases} \mathbb{Z}_2 \cong \langle -1_{\mathrm{Spin}(n)} \rangle & (n \text{ is odd}), \\ \mathbb{Z}_4 \cong \langle \mathfrak{z} \rangle & (n \equiv 2 \bmod 4), \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle -1_{\mathrm{Spin}(n)} \rangle \times \langle \mathfrak{z} \rangle & (n \equiv 0 \bmod 4). \end{cases} \tag{6.44}$$

The Co_1-action on $V^{f\natural}$

Take the Conway group $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24}) \subset \mathrm{SO}(24)$.

Proposition 6.3 ([DMC14, Prop. 3.1]). *There is a lift $\hat{G} \subset \mathrm{Spin}(24)$ of $\mathrm{Co}_0 \subset \mathrm{SO}(24)$ such that $\hat{G}$ is isomorphic to Co_0, and such a lift is unique.*

Proof. The existence of $\hat{G} \cong \mathrm{Co}_0$ is due to the fact that the Schur multiplier $H_2(\mathrm{Co}_0; \mathbb{Z})$ is trivial [Gri74, §9], and Co_0 is a perfect group[37] $H_1(\mathrm{Co}_0; \mathbb{Z}) = 0$.

We can see the uniqueness as follows. According to the Co_1 page of [WWT+], Co_0 is generated by two elements A, B such that[38] $(ABABABABBABABBABB)^{33} = 1$ and $B^3 = 1$. Since these relations contain odd numbers of A and B respectively, these relations determine which of the two lifts $\pm\hat{A} \in \mathrm{Spin}(24)$ of A is the generator $\hat{A}$ of $\hat{G} \subset \mathrm{Spin}(24)$, and which of $\pm\hat{B}$ is the generator as well. □

We take an orthonormal basis $\psi_1, \ldots, \psi_{24}$ of $\Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{R}$, and introduce the polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$ as in (6.17). This determines which of the two lifts of $-1_{\mathrm{SO}(24)} \in \mathrm{SO}(24)$ is $\mathfrak{z} \in \mathrm{Spin}(24)$; the other one is $-\mathfrak{z}$. Since $-1_{\mathrm{SO}(24)} \in \mathrm{Co}_0$, one of $\pm\mathfrak{z}$ is contained in $\hat{G}$, and the other is not. Recalling (6.31) $\mathfrak{z} = \psi_1 \cdots \psi_{24}$, we can rename ψ_1 and ψ_2 so that $\mathfrak{z} \in \hat{G}$. In this way, we always assume that we take the orthonormal basis of $\Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{R}$ and the polarization such that $\mathfrak{z} \in \hat{G}$.

Since the action of $\mathfrak{z}$ on $V^{f\natural}$ is trivial, the action of $\hat{G} \subset \mathrm{Spin}(24)$ on $V^{f\natural}$ reduces to the action by the quotient $\hat{G}/\langle\mathfrak{z}\rangle \cong \mathrm{Co}_1 \subset \mathrm{SemiSpin}(24)$.

We will also write $\hat{G}$ as Co_0 when there is no risk of confusion.

$$\begin{array}{ccc} & \mathrm{Co}_0 \subset \mathrm{Spin}(24) \curvearrowright A(\mathfrak{a}) \oplus A(\mathfrak{a})_{\mathrm{tw}} & \\ \swarrow^{/\langle\mathfrak{z}\rangle} & & \searrow^{\cong} \\ \mathrm{Co}_1 \subset \mathrm{SemiSpin}(24) \curvearrowright V^{f\natural} & & \mathrm{Co}_0 \subset \mathrm{SO}(24) \curvearrowright A(\mathfrak{a}) \end{array} \tag{6.46}$$

Remark 6.4. On $V^{s\natural}$, the action of $-\mathfrak{z}$ is trivial, but the action of $\mathfrak{z}$ is not. So we can say that the action of $\mathrm{Spin}(24)$ reduces to the action of $\mathrm{Spin}(24)/\langle -\mathfrak{z}\rangle \cong \mathrm{SemiSpin}(24)$, but since $-\mathfrak{z} \notin \mathrm{Co}_0 \subset \mathrm{Spin}(24)$, the action of Co_0 does not reduce to the action of Co_1. In particular, $V^{f\natural}$ and $V^{s\natural}$ are isomorphic as VOSAs, but they are not as Co_0-modules. *(Remark ends.)*

[37] A group G is said to be *perfect* if its abelianization $G^{\mathrm{ab}} := G/[G,G]$ is trivial. Since the only normal subgroups of Co_0 are the trivial ones 1, Co_0, and its center $\mathbb{Z}_2$ satisfying $\mathrm{Co}_0/\mathbb{Z}_2 \cong \mathrm{Co}_1$, it is obvious that Co_0 is perfect. Using $G^{\mathrm{ab}} \cong H_1(G;\mathbb{Z})$, we have $H_1(\mathrm{Co}_0;\mathbb{Z}) = 0$. By applying $H_1(\mathrm{Co}_0;\mathbb{Z}) = 0$ and the trivial Schur multiplier $H_2(\mathrm{Co}_0;\mathbb{Z}) = 0$ to the universal coefficient theorem, which states $0 \to \mathrm{Ext}_{\mathbb{Z}}(H_1(G;\mathbb{Z});A) \to H^2(G;A) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(G;\mathbb{Z});A) \to 0$ is exact, we have $H^2(\mathrm{Co}_0;\mathbb{Z}_2) = 0$. This means any extension of Co_0 by $\mathbb{Z}_2$ splits, and hence there is a group-homomorphic lift of $\mathrm{Co}_0 \subset \mathrm{SO}(24)$ to $\mathrm{Spin}(24)$.

[38] The convention of [WWT+] is the right action, so if we use A, B introduced in (2.46, 2.47) which act on column vectors in $\mathbb{Z}^{24}$ from the left, we have to reverse the order of the multiplications in these relations. Note that the Version 3 webpage https://brauer.maths.qmul.ac.uk/Atlas/v3/spor/Co1/ is not maintained, and there is a typo which lacks the last B as "$(ABABABABBABABBAB)^{33} = 1$," when accessed November 9th, 2025. Instead, the Version 2 webpage https://brauer.maths.qmul.ac.uk/Atlas/spor/Co1/ is maintained when accessed, and presents the correct relation.

6.3 Conway Moonshine

Recall that $V^{f\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^0_{\mathrm{tw}}$, and states in $A(\mathfrak{a})^0$ have integer weights, whereas those in $A(\mathfrak{a})^0_{\mathrm{tw}}$ have half-integer weights. In particular, the weight-$\frac{3}{2}$ subspace of $A(\mathfrak{a})^0_{\mathrm{tw}}$ is

$$\mathbf{2}^{\mathbf{11}}_{+} := \mathrm{Span}_{\mathbb{C}}\{(\text{even number of } \Psi_i(0)\text{'s})|0\rangle_{\mathrm{R}}\} \subset A(\mathfrak{a})^0_{\mathrm{tw}}. \tag{6.47}$$

This $\mathbf{2}^{\mathbf{11}}_{+}$ and

$$\mathbf{2}^{\mathbf{11}}_{-} := \mathrm{Span}_{\mathbb{C}}\{(\text{odd number of } \Psi_i(0)\text{'s})|0\rangle_{\mathrm{R}}\} \subset A(\mathfrak{a})^1_{\mathrm{tw}}, \tag{6.48}$$

are called the *chiral representations* or the *Weyl spinor representations* of $\mathrm{Spin}(24)$.

We have seen that $\mathrm{Co}_1 \subset \mathrm{SemiSpin}(24)$ acts on $V^{f\natural}$. As a Co_1-module, this weight-$\frac{3}{2}$ subspace $\mathbf{2}^{\mathbf{11}}_{+}$ decomposes into irreducible modules as [DMC14, proof of Prop. 4.4]

$$\mathbf{2}^{\mathbf{11}}_{+} = \mathbf{1} \oplus \mathbf{276} \oplus \mathbf{1771}, \tag{6.49}$$

where $\mathbf{1}$ is the trivial representation, whose states are invariant under the Co_1-action. Moreover, a suitably scaled basis τ of this Co_1-invariant subspace $\mathbf{1}$ constitutes the supercurrent of an $\mathcal{N} = 1$ superconformal algebra (4.22, 4.23) in $V^{f\natural}$ [DMC14, Prop. 4.4].

Conversely, it is proved [Dun07, Thm. 4.11] that the group $\mathrm{Aut}_{\mathcal{N}=1}(V^{f\natural})$ of all the automorphisms of the VOSA $V^{f\natural}$ preserving this $\mathcal{N} = 1$ structure is precisely isomorphic to the Conway group Co_1. To summarize,

Theorem 6.5 ([Dun07, Thm. 4.11])**.** *The VOSA $V^{f\natural}$ admits an $\mathcal{N} = 1$ structure such that $\mathrm{Aut}_{\mathcal{N}=1}(V^{f\natural}) \cong \mathrm{Co}_1$.*

Furthermore, its uniqueness is also known as follows, although we do not go into the details.

Theorem 6.6 ([DMC14, Thm. 4.5])**.** *Let V be a self-dual C_2-cofinite rational VOSA of CFT type, with central charge 12 and without weight-$\frac{1}{2}$ states $V_{\frac{1}{2}} = 0$. Then V is isomorphic to $V^{f\natural}$ as a VOSA.*

In particular, V in this Theorem 6.6 admits an $\mathcal{N} = 1$ structure such that $\mathrm{Aut}_{\mathcal{N}=1}(V) \cong \mathrm{Co}_1$.

Of course, we can see the appearance of representation dimensions of Co_1 in the coefficients of the partition function of $V^{f\natural}$ [Dun07, Eq. (1.0.2)]

$$\mathrm{Tr}_{V^{f\natural}} q^{L_0 - \frac{12}{24}} = \frac{\Theta_{E_8}(\tau)\eta(\tau)^8}{\eta(\tau/2)^8\eta(2\tau)^8} - 8 \tag{6.50}$$

$$= q^{-\frac{1}{2}}(1 + 276q + 2^{11}q^{\frac{3}{2}} + 11202q^4 + 49152q^{\frac{5}{2}} + \cdots), \tag{6.51}$$

where 276 is the smallest nontrivial irreducible representation dimension of Co_1, $2^{11} = 1+276+1771$ as in (6.49), $11202 = 1+276+299+1771+8855$, and so on. Since the Co_1-action preserves the $\mathcal{N} = 1$ superconformal algebra, we will still see the representation dimensions of Co_1 even if we decompose the partition function into $\mathcal{N} = 1$ superconformal algebra characters.

Now, we can see that the three important sporadic groups $\mathbb{M}$, Co_1, and M_{24} can be realized as automorphism groups of the following objects.

Aut	algebraic object	VOA
$\mathbb{M}$	the Griess algebra $B^\natural$	the (bosonic) VOA $V^\natural$
Co_1	the (quotient of) Leech lattice $\Lambda_{24}/\{\pm 1\}$	the $\mathcal{N}=1$ VOSA $V^{f\natural}$
M_{24}	the Golay code G_{24}	?

(6.52)

Remark 6.7. Among the algebraic objects listed above (6.52), we have already mentioned the uniqueness of G_{24}, Λ_{24}, and $V^{f\natural}$. The Golay code G_{24} is the unique binary code of length 24, dimension 12, and minimal nonzero weight 8 (Section 2.2.1). The Leech lattice Λ_{24} is the unique even self-dual lattice of rank 24 and without vectors of squared length 2 (Section 2.3.1). Duncan's module $V^{f\natural}$ is the unique VOSA of central charge 12 and without weight-$\frac{1}{2}$ states (Theorem 6.6).

As mentioned in Section 2.4, the Griess algebra $B^\natural$ is the weight-2 subspace $(V^\natural)_2$ of the monster VOA $V^\natural$. The uniqueness of the monster VOA $V^\natural$ as a VOA of central charge 24 and without weight-1 states is still an open problem, called the *Frenkel–Lepowsky–Meurman uniqueness conjecture*. See [DGL07] for a partial result. One interest thing is that VOAs of central charge 24 with weight-1 states $V_1 \neq 0$ are known to be uniquely determined from the structure of the weight-1 subspace V_1, and already classified as 70 VOAs; see for example [vELMS21, Lam23]. Therefore, if the uniqueness of the monster VOA is proved, then the proof of the list of 71 VOAs of central charge 24 conjectured by Schellekens [Sch93] will be completed. *(Remark ends.)*

7 Mathieu Moonshine

A K3 surface is a complex-two-dimensional Calabi–Yau manifold. More precisely,[39] a complex-two-dimensional compact Kähler manifold X is said to be a *K3 surface* if its canonical bundle is trivial, and it has the Hodge number $h^{0,1}(X) = 0$. It is known that any K3 surfaces are diffeomorphic, but their isomorphism classes as complex manifolds are not unique. The second integral cohomology[40] $H^2(X;\mathbb{Z})$ of a K3 surface together with the symmetric bilinear form defined by the cup product is a lattice of signature $(3,19)$ and isometric to[41] $\mathrm{II}_{1,1}^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. The isomorphism classes of K3 surfaces are parametrized by the real-two-dimensional plane in $H^2(X;\mathbb{R})$, specified by the nowhere-vanishing holomorphic 2-form ω on X. The space of all the isomorphism classes of K3 surfaces is called the *moduli space* of K3 surfaces. See for example [Kon18, Kon20, Asp96] for more details.

We can consider the supersymmetric non-linear sigma model (full or non-chiral two-dimensional SCFT) with its target space being a K3 surface X. We call it a *K3 sigma model* for short. It has $\dim_{\mathbb{R}} X$ bosons and fermions, so the central charge is $c = \tilde{c} = 4 \times (1 + \frac{1}{2}) = 6$. Thanks to the hyperKähler structure of the K3 surface, it has $\mathcal{N} = (4,4)$ supersymmetries. Hence, the elliptic genus $Z_{\mathrm{ell}}^{(\mathrm{K3})}(\tau, z)$ (see Section 3.4) of a K3 sigma model is a weak Jacobi form of weight 0 and index 1. From Theorem 3.4, such a weak Jacobi form is $\varphi_{0,1}(\tau, z)$ up to scalar multiplication, and it was calculated in [EOTY89] as

$$Z_{\mathrm{ell}}^{(\mathrm{K3})}(\tau, z) = 2\varphi_{0,1}(\tau, z) = 8\left(\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right). \tag{7.1}$$

This elliptic genus is independent of the choice of the target K3 surface, or equivalently, constant over the moduli space.

Recall that the elliptic genus is a $(-1)^F$-inserted partition function of the left-moving R sectors coupling to the right-moving Ramond vacua. Since the left-moving part of a K3 sigma model has the $\mathcal{N} = 4$ supersymmetry, we can decompose the elliptic genus $Z_{\mathrm{ell}}^{(\mathrm{K3})}(\tau, z)$ into the $(-1)^F$-inserted characters of irreducible Ramond representations of the $\mathcal{N} = 4$ superconformal algebra of central charge $c = 6$. Such irreducible characters are labeled by the conformal weight h and the spin l (with respect to the $\widehat{\mathfrak{su}}(2)_{\frac{c}{6}}$ algebra consisting of the three supercurrents) of the highest weight state, and classified into

- the massless characters $\mathrm{ch}_{(0)h,l}^{\mathcal{N}=4}(\tau, z)$ with $h = \frac{1}{4}$ and $l = 0, \frac{1}{2}$,

[39]There are several different definitions of a Calabi–Yau manifold. If we define it as a compact Kähler manifold X with trivial canonical bundle, then two-dimensional Calabi–Yau manifolds are only complex-2-tori and K3 surfaces. Between them, K3 surfaces can be specified by the condition $h^{0,1}(X) = 0$ as in the main text, or by the condition that the holonomy group is $\mathrm{SU}(2)$. (The holonomy of a torus is trivial.) So some literature uses the definition of a Calabi–Yau manifold containing the condition that the holonomy is $\mathrm{SU}(n)$ where n is the complex dimension.

[40]In the notation of footnote 1, $H^2(X;\mathbb{Z})$ in the main text here should be written as $H^2_{\mathrm{top}}(X;\mathbb{Z})$.

[41]$\mathrm{II}_{1,1}$ is the even self-dual lattice of signature $(1,1)$. It can be written as $\mathrm{II}_{1,1} = (\mathbb{Z}^2, q)$ where the symmetric bilinear form q has the matrix form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $-E_8$ denotes the E_8 lattice whose symmetric bilinear form is multiplied by -1, and hence of signature $(0,8)$.

- the massive characters $\mathrm{ch}^{\mathcal{N}=4}_{h,l}(\tau,z)$ with $h=\frac{1}{4}+\mathbb{Z}_{>0}$ and $l=\frac{1}{2}$.

Their explicit formulae are derived in [ET88] (also reviewed in [EOT10]) as

$$\mathrm{ch}^{\mathcal{N}=4}_{(0)\frac{1}{4},0}(\tau,z)=\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\mu(\tau,z),\quad \mu(\tau,z):=\frac{-\sqrt{-1}y^{1/2}}{\theta_1(\tau,z)}\sum_{l\in\mathbb{Z}}\frac{(-1)^l y^l q^{l(l+1)/2}}{1-yq^l}, \tag{7.2}$$

$$\mathrm{ch}^{\mathcal{N}=4}_{h,\frac{1}{2}}(\tau,z)=\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}q^{h-\frac{3}{8}}. \tag{7.3}$$

$\mathrm{ch}^{\mathcal{N}=4}_{(0)\frac{1}{4},\frac{1}{2}}(\tau,z)$ can be derived from

$$\mathrm{ch}^{\mathcal{N}=4}_{\frac{1}{4},\frac{1}{2}}(\tau,z)=2\,\mathrm{ch}^{\mathcal{N}=4}_{(0)\frac{1}{4},0}(\tau,z)+\mathrm{ch}^{\mathcal{N}=4}_{(0)\frac{1}{4},\frac{1}{2}}(\tau,z), \tag{7.4}$$

where we allowed $h=\frac{1}{4}$ for the massive character (7.3) as $\mathrm{ch}^{\mathcal{N}=4}_{\frac{1}{4},\frac{1}{2}}(\tau,z)=\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}q^{-\frac{1}{8}}$.

As a result, the K3 elliptic genus $Z^{(\mathrm{K3})}_{\mathrm{ell}}(\tau,z)$ is decomposed as [Oog89, EOT10]

$$Z^{(\mathrm{K3})}_{\mathrm{ell}}(\tau,z)=20\,\mathrm{ch}^{\mathcal{N}=4}_{(0)\frac{1}{4},0}(\tau,z)-2\,\mathrm{ch}^{\mathcal{N}=4}_{(0)\frac{1}{4},\frac{1}{2}}(\tau,z)+2\sum_{n=1}^{\infty}A_n\mathrm{ch}^{\mathcal{N}=4}_{n+\frac{1}{4},\frac{1}{2}}(\tau,z) \tag{7.5}$$

$$=24\,\mathrm{ch}^{\mathcal{N}=4}_{(0)\frac{1}{4},0}(\tau,z)-2\,\mathrm{ch}^{\mathcal{N}=4}_{\frac{1}{4},\frac{1}{2}}(\tau,z)+2\sum_{n=1}^{\infty}A_n\mathrm{ch}^{\mathcal{N}=4}_{n+\frac{1}{4},\frac{1}{2}}(\tau,z), \tag{7.6}$$

where

n	1	2	3	4	5	6	7	8	9	$\cdots$
A_n	45	231	770	2277	5796	13915	30843	65550	132825	$\cdots$

(7.7)

In 2010, Eguchi, Ooguri, and Tachikawa [EOT10] observed that these coefficients can be written as simple sums of irreducible representation dimensions of the largest Mathieu group M_{24}. The 26 irreducible representation dimensions of M_{24} are, from the smallest one,

$$\begin{aligned}&1,23,45,45,231,231,252,253,483,770,770,\\&990,990,1035,1035,1035,1265,1771,2024,\\&2277,3312,3520,5313,5796,5544,10395.\end{aligned}$$

So $A_1,\dots,A_5$ are directly the irreducible representation dimensions of M_{24}, and

$$A_6=3520+10395, \tag{7.8}$$

$$A_7=1771+2024+5313+5544+5796+10395. \tag{7.9}$$

This observation by [EOT10] triggered the effort to construct a coherent theory analogous to the monstrous moonshine, so-called the *(K3) Mathieu moonshine*. Recall from Section 5.1 that the McKay–Thompson series $J_g(\tau)$ is the character of $g\in\mathbb{M}$ represented on the moonshine module $V^\natural$, and in the language of physics, it is a twisted partition function twisted in the temporal direction. The analogue of the McKay–Thompson series, that is, the character $\phi_g(\tau, z)$ of $g \in M_{24}$ instead of $1_{M_{24}}$ in the elliptic genus $Z^{(\mathrm{K3})}_{\mathrm{ell}}(\tau, z)$ was calculated in [Che10, GHV10a, GHV10b, EH10], and called the *twining elliptic genus* in [GHV10a, GHV10b]. In addition, the existence of an M_{24}-module which reproduces these twining elliptic genera as its characters was shown in [Gan12]. The twisted partition functions twisted in both spatial and temporal directions, called the *twisted twining elliptic genera*, were also calculated in [GPRV12]. Remarkably, their modular transformation properties fit into the theory of anomaly of orbifold (Section F.1.2), and controlled by a 3-cocycle in $H^3(M_{24}; \mathrm{U}(1))$. This suggests that there exists an underlying VOA with M_{24} symmetry and the elliptic genus $Z^{(\mathrm{K3})}_{\mathrm{ell}}(\tau, z)$.

In spite of these pieces of evidence, such an underlying VOA has not yet been found, and many mysterious aspects still remain. First, the geometric symmetry of any K3 surface is smaller than the subgroup M_{23} of M_{24}. More precisely, the following theorem is known.

Theorem 7.1 (Mukai [Muk88])**.** *For a finite group G, the followings are equivalent.*

(1) There exists a K3 surface X such that its automorphism group $\mathrm{Aut}_\omega(X)$ preserving the nowhere-vanishing holomorphic 2-form ω on X contains G as a subgroup.

(2) G is a subgroup of M_{23}, and as a subgroup of M_{24} acting on 24 points Ω_{24}, the action of G on Ω_{24} has at least 5 orbits in Ω_{24}.

In the same spirit, the automorphism groups of K3 sigma models preserving the $\mathcal{N} = (4, 4)$ superconformal algebra were studied in [GHV11], through a strategy similar to Kondō's proof [Kon98] of Mukai's Theorem 7.1. They all turned out to be subgroups of the Conway group Co_1, but none of them are M_{24}, or contain M_{24} as a subgroup. Instead, some of them are proper subgroups of M_{24}, and the others are not even subgroups of M_{24}.

One proposal to resolve this problem was provided in [TW11, TW13a, TW13b]. Since the elliptic genus $Z^{(\mathrm{K3})}_{\mathrm{ell}}(\tau, z)$ is constant over the moduli space of K3 surfaces, they glued the automorphism groups of the K3 surfaces at three different points in the moduli space together, and succeeded in constructing the action of a maximal subgroup $2^4 : A_8$ called the octad subgroup of M_{24} on the first massive part $\mathbf{45} \oplus \overline{\mathbf{45}}$ corresponding to the term $2A_1 \mathrm{ch}^{\mathcal{N}=4}_{\frac{5}{4},\frac{1}{2}}(\tau, z)$ in $Z^{(\mathrm{K3})}_{\mathrm{ell}}(\tau, z)$ (7.5). This idea is referred to as *symmetry-surfing*. Their action of $2^4 : A_8$ in fact coincides with the M_{24}-action restricted to $2^4 : A_8$ on the irreducible representation $\mathbf{45}$ of M_{24}. Its application to the higher-weight parts is discussed in [GKP16].

Another possible way to circumvent the situation is to loosen the condition on the automorphism group that it should preserve the $\mathcal{N} = (4, 4)$ superconformal algebra. In fact, we only need the $\mathcal{N} = (4, 1)$ supersymmetry to consider the elliptic genus and its decomposition into the irreducible characters of the left-moving $\mathcal{N} = 4$ superconformal algebra, so the automorphism group preserving only the $\mathcal{N} = (4, 1)$ supersymmetry is still useful. For example, a K3 sigma model with $2^8 : M_{20}$ symmetry, which is one of the maximal symmetries given in [GHV11] (and not a subgroup of M_{24}), was constructed in [GTVW13]. This K3 sigma model was further studied in relation to quantum error-correcting codes in [HM20], and they also showed that the automorphism group preserving the $\mathcal{N}=(4,1)$ superconformal algebra is indeed larger than $2^8 : M_{20}$, although it does not contain M_{24}. Of course, we may seek an $\mathcal{N}=(4,1)$ SCFT with M_{24} symmetry and elliptic genus $Z_{\text{ell}}^{(\text{K3})}(\tau, z)$ (so the left-moving central charge should be 6), other than K3 sigma models.

As another approach, the relation to Duncan's module (Section 6) is also being explored. As a naive observation, since a K3 sigma model has central charge $(c,\tilde{c})=(6,6)$, if we "reflect" the right-moving part to the left-moving part, then we obtain a chiral SCFT with central charge $c=12$, which is the same central charge as Duncan's module. In fact, under an appropriate procedure called *reflection*, it was shown [DMC15, TW17] that the reflection of the specific K3 sigma model constructed in [GTVW13] is isomorphic to Duncan's module as a VOSA. As reviewed in Section 6.3, Duncan's module admits an $\mathcal{N}=1$ superconformal algebra, and the automorphism group preserving it is the Conway group Co_1, which contains M_{24} as a subgroup. In addition, it is known that Duncan's module also admits $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal algebras [CDD$^+$14] (see also Section 3.4). However, the automorphism group of Duncan's module preserving its $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal algebras are only M_{23} and M_{22}, respectively [CDD$^+$14]. It was also pointed out in [Gan12] that if the character of M_{24} appearing in the elliptic genus $Z_{\text{ell}}^{(\text{K3})}(\tau, z)$ is a restriction of a virtual character (a signed sum of irreducible characters) of some representation of Co_1 or Co_0, then such a representation of Co_1 or Co_0 must have an excessively large dimension. (For example, the smallest virtual representation of Co_0 which restricts to the representation $\mathbf{45}\oplus\overline{\mathbf{45}}$ of M_{24} has dimension over 100 billion.)

Lastly, we mention that, as a generalization of the Mathieu moonshine, the *umbral moonshine* was proposed in [CDH12, CDH13]. It relates a certain quotient $\mathrm{Aut}(L)/\mathrm{Weyl}(L)$ of the isometry group to a mock modular form, for each Niemeier lattice L except for the Leech lattice Λ_{24}, and contains the K3 Mathieu moonshine as a case of $L=(A_1)^{24}$. The result of [Gan12] for the K3 Mathieu moonshine was generalized to the other cases of the umbral moonshine by [DGO15]. For more on Mathieu moonshine and its recent developments, see for example [Tac12, DGO14, Car17, HHP22, JF20] and references therein.

Part III
Appendices

A Group Extension

In this Section A, we give an elementary introduction to group extensions and a few related topics. In Section A.1, we review basic notions of group extensions following [Bro82, Ch. IV], and see that the equivalence classes of group extensions of a group G by a G-module N are in one-to-one correspondence with the cohomology classes of the second group cohomology $H^2(G, N)$. In the next Section A.2, we focus on central extensions of abelian groups, and establish another one-to-one correspondence with commutator maps in the case of free abelian groups, following [FLM88, §5.2]. When a central extension of a free abelian group L is by $\mathbb{Z}_2$, we can further consider additional information, quadratic forms, and corresponding extensions of $L/2L$ by $\mathbb{Z}_2$. This is of particular importance in the construction of the $\mathbb{Z}_2$-twisted sector of a lattice VOA, so reviewed in Section A.3. The last Section A.4 is a review of a certain theorem [FLM88, Prop. 5.4.1] on the automorphism group of a central extension, which plays a vital role in the description of the automorphism group of a lattice VOA in Section 5.2.3.

A.1 Group Extensions and Group Cohomology

An *extension* of a group G by a group N is a short exact sequence of groups and homomorphisms

$$1 \to N \xrightarrow{i} \hat{G} \xrightarrow{\pi} G \to 1. \tag{A.1}$$

(Be aware that some literature calls it an extension *of* N *by* G.) When there is no risk of confusion in the homomorphisms constituting the short exact sequence, we only say that $\hat{G}$ is an extension of G by N. (But note that the equivalence of group extension, defined later, classifies not only $\hat{G}$, but also the whole short exact sequence.) We also use the notation $N.G$ for any extension of G by N. Since $i(N)$ is the kernel of π, N can be regarded as a normal subgroup of $\hat{G}$, and the quotient group $\hat{G}/i(N)$ is isomorphic to G.

Let us take a set-theoretical section (not necessarily a group homomorphism) $s : G \to \hat{G}$. Since $s(g)s(g')s(gg')^{-1} \in \ker \pi = i(N)$ for $g, g' \in G$, we can define a function $\epsilon : G \times G \to N$ which measures how s failures to be a homomorphism by

$$s(g)s(g') = i(\epsilon(g, g'))s(gg'). \tag{A.2}$$

To look into the structure of $\hat{G}$, we use the bijection $N \times G \to \hat{G}\,; (n, g) \mapsto i(n)s(g)$. Recalling that $i(N)$ is a normal subgroup of $\hat{G}$, we can define the action of $h \in \hat{G}$ on N by

$$\hat{\varphi}_h(n) = i^{-1}(hi(n)h^{-1}), \tag{A.3}$$

and then the multiplication law of $\hat{G}$ can be calculated as

$$i(n)s(g) \cdot i(n')s(g') = i(n \cdot \hat{\varphi}_{s(g)}(n') \cdot \epsilon(g, g'))s(gg'). \tag{A.4}$$

This multiplication law (i.e. the group structure of $\hat{G}$) becomes simpler in the following cases:

- If we can take a section s which is a group homomorphism (i.e. the short exact sequence (A.1) *splits*), then the cocycle becomes trivial $\epsilon(g, g') = 1_N$. In addition, we can define a G-action σ on N by $\sigma_g := \hat{\varphi}_{s(g)}$. As a result, $\hat{G}$ is isomorphic to the semidirect product $N \rtimes_\sigma G$, and it is also denoted by $N : G$. In particular, G can be regarded as a subgroup of $\hat{G}$ by s.
  - Moreover, if the G-action σ on N is trivial, then $\hat{G}$ is isomorphic to the direct product $N \times G$.
- If N is abelian, then for a given $g \in G$, any element $h \in \hat{G}$ such that $\pi(h) = g$ defines the same action $\hat{\varphi}_h$ on N, so the $\hat{G}$-action $\hat{\varphi}$ reduces to a G-action φ on N.
  - Moreover, if (and only if) $i(N)$ is in the center of $\hat{G}$, the G-action φ becomes trivial. In this case, $\hat{G}$ is called a *central extension*.

We also use the notation $N^{\cdot}G$ for an extension $\hat{G}$ such that the short exact sequence (A.1) does not split.

In the following, we assume N is abelian. An abelian group on which a group G acts is called a *G-module*. Since there is the G-action φ on N, N is a G-module.

From the associativity $(s(g)s(g'))s(g'') = s(g)(s(g')s(g''))$, the function ϵ turns out to be an N-valued 2-cocycle:

$$\epsilon(g, g')\epsilon(gg', g'') = \varphi_g(\epsilon(g', g''))\epsilon(g, g'g''). \tag{A.5}$$

If we take another section s', then the cocycle ϵ' for it differs from ϵ by only coboundary. To see it, we define $\zeta : G \to N$ as the difference of $s(g)$ and $s'(g)$:

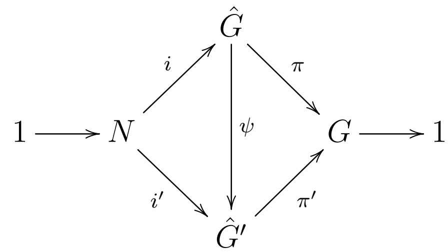
$$s'(g) = i(\zeta(g))s(g). \tag{A.6}$$

Then we can calculate ϵ' based on the definition (A.2) as

$$\epsilon'(g, g') = \epsilon(g, g')\varphi_g(\zeta(g'))\zeta(gg')^{-1}\zeta(g), \tag{A.7}$$

which shows that ϵ' and ϵ differ by the coboundary $d\zeta$. Conversely, if a cocycle ϵ' is cohomologous to the cocycle ϵ of the section s as $\epsilon' = \epsilon d\zeta$, then ϵ' is the cocycle of the section s' such that $s'(g) = i(\zeta(g))s(g)$.

Two extensions $1 \to N \xrightarrow{i} \hat{G} \xrightarrow{\pi} G \to 1$ and $1 \to N \xrightarrow{i'} \hat{G}' \xrightarrow{\pi'} G \to 1$ are said to be *equivalent* if there exists a homomorphism $\psi : \hat{G} \to \hat{G}'$ such that the diagram



(A.8)

commutes. Such ψ is an isomorphism by the short five lemma.[42] The difference of the cocycle ϵ for a section $s : G \to \hat{G}$ and the cocycle ϵ' for a section $s' : G \to \hat{G}'$ is again a coboundary, because ϵ is also the cocycle for the section $\psi \circ s : G \to \hat{G}'$. Note that two extensions can be non-equivalent even if $\hat{G}$ and $\hat{G}'$ are isomorphic as groups.[43] In this sense, the equivalence of group extension classifies not only $\hat{G}$, but also how N and G are incorporated in it.

So far, for a given G-module N, we have established a map

$$\flat : \left\{ \begin{array}{c} \text{extensions of } G \text{ by } N \\ \text{compatible with the action } G \curvearrowright N \end{array} \right\} / \text{equivalence} \to H^2(G, N). \tag{A.9}$$

We can construct a map $\sharp$ in the inverse direction of (A.9) as follows. For a given G-module N and a 2-cocycle $\epsilon : G \times G \to N$, we can construct a group extension $\hat{G}_\epsilon$ as a set $N \times G$ with the multiplication[44]

$$(n, g) \cdot (n', g') = (n \cdot g(n') \cdot \epsilon(g, g'), gg'), \tag{A.10}$$

together with group homomorphisms

$$i : N \to \hat{G}_\epsilon \,; n \mapsto (n \cdot \epsilon(1_G, 1_G)^{-1}, 1_G), \tag{A.11}$$

$$\pi : \hat{G}_\epsilon \to G \,; (n, g) \mapsto g, \tag{A.12}$$

constituting the short exact sequence.

---

[42]For the short five lemma to be applied, ψ should be a homomorphism, not just a map. In fact, for any two extensions $\hat{G}, \hat{G}'$ of G by N, there exists a map ψ such that the diagram (A.8) commutes (e.g. just $i(n)s(g) \mapsto i'(n)s'(g)$ for sections $s : G \to \hat{G}$ and $s' : G \to \hat{G}'$ normalized as in (A.21)), but it is of course not an isomorphism in general.

[43]For example, for additive groups $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $N = \mathbb{Z}_2$, (i) $\hat{G} = \mathbb{Z}_4 \times \mathbb{Z}_2$ with $i(1) = (2, 0)$ and $\pi((a, b)) = (a, b) \mod 2$, and (ii) $\hat{G} = \mathbb{Z}_4 \times \mathbb{Z}_2$ with $i(1) = (2, 0)$ and $\pi((a, b)) = (b, a) \mod 2$ are inequivalent extensions.

[44]Under this multiplication, the identity element of $\hat{G}_\epsilon$ is $(\epsilon(1_G, 1_G)^{-1}, 1_G)$, and the inverse of (n, g) is $(\epsilon(1_G, 1_G)^{-1} g^{-1}(n^{-1}) \epsilon(g^{-1}, g)^{-1}, g^{-1})$. We can check them by using $\epsilon(1_G, g) = \epsilon(1_G, 1_G)$, $\epsilon(g, 1_G) = g(\epsilon(1_G, 1_G))$, and $\epsilon(g, g^{-1})\epsilon(1_G, g) = g(\epsilon(g^{-1}, g))\epsilon(g, 1_G)$, which all follow from the cocycle condition (A.5).

For another 2-cocycle ϵ', if it differs from ϵ by the coboundary $d\zeta$ as in (A.7), then

$$\psi : \hat{G}_\epsilon \to \hat{G}_{\epsilon'} \,;\, (n,g) \mapsto (n\zeta(g)^{-1}, g) \tag{A.13}$$

defines an homomorphism to show the equivalence of $\hat{G}_\epsilon$ and $\hat{G}_{\epsilon'}$. Therefore, we have established the map $\sharp$ in the inverse direction of (A.9).

In fact, $\flat$ and $\sharp$ are the inverse of each other. To see it, we first note that any 2-cocycle ϵ satisfies

$$\epsilon(1_G, g) = \epsilon(1_G, 1_G) \quad \text{for any } g \in G, \tag{A.14}$$

which follow from the cocycle condition (A.5). We also have

$$s(1_G) = i(\epsilon(1_G, 1_G)), \tag{A.15}$$

from (A.2), for any section $s : G \to \hat{G}$ of an extension and the cocycle ϵ for it.

To see $\sharp \circ \flat$ is an identity map, for a given extension $\hat{G}$, take a section $s : G \to \hat{G}$ and construct the extension $\hat{G}_\epsilon$ from the cocycle ϵ for s. Then it is equivalent to the original extension $\hat{G}$ by

$$\psi : \hat{G}_\epsilon \to \hat{G} \,;\, (n,g) \mapsto i(n)s(g). \tag{A.16}$$

The most nontrivial part is $\psi \circ (i \text{ for } \hat{G}_\epsilon) = (i \text{ for } \hat{G})$, which follows from

$$\psi((n\epsilon(1_G,1_G)^{-1}, 1_G)) = i(n\epsilon(1_G,1_G)^{-1})s(1_G) \overset{\text{(A.15)}}{=} i(n). \tag{A.17}$$

To see $\flat \circ \sharp$ is an identity map, starting from a given cocycle ϵ, construct the extension $\hat{G}_\epsilon$ from it. Then the section $s : G \to \hat{G}_\epsilon \,;\, g \mapsto (1_N, g)$ gives back the cocycle ϵ because

$$s(g)s(g') = (\epsilon(g,g'), gg') = (\epsilon(g,g')\epsilon(1_G, gg')^{-1}, 1_G) \cdot (1_N, gg') \tag{A.18}$$

$$\overset{\text{(A.14)}}{=} (\epsilon(g,g')\epsilon(1_G,1_G)^{-1}, 1_G) \cdot (1_N, gg') = i(\epsilon(g,g'))s(gg'). \tag{A.19}$$

To summarize the above discussions, we finally obtained the following theorem.

Theorem A.1. *For a group G and a G-module N, there exists a one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{extensions of } G \text{ by } N \\ \text{compatible with the action } G \curvearrowright N \end{array} \right\} / \text{equivalence} \underset{\sharp}{\overset{\flat}{\rightleftarrows}} H^2(G,N). \tag{A.20}$$

Lastly, we mention the normalization of sections and cocycles. If we take a section $s : G \to \hat{G}$ satisfying the normalization condition

$$s(1_G) = 1_{\hat{G}}, \tag{A.21}$$

then the cocycle ϵ for it satisfies the normalization condition

$$\epsilon(1_G, 1_G) = 1_N. \tag{A.22}$$

Therefore, by Theorem A.1, any cohomology class in $H^2(G,N)$ has at least one normalized cocycle. In fact, we can construct it explicitly; for any cocycle ϵ', if we define $\zeta(g) := \epsilon'(g,g)^{-1}$, then the modified cocycle $\epsilon := \epsilon' \cdot d\zeta$ satisfies the normalization (A.22). Hence, in many cases, we can restrict our attention to the normalized sections and cocycles, without loss of generality.

A.2 Central Extensions of Free Abelian Group and Commutator Maps

Let G be an abelian group and $1 \to N \xrightarrow{i} \hat{G} \xrightarrow{\pi} G \to 1$ be a central extension of G, that is, N is a G-module by the trivial G-action. Now, any commutator $[h, h'] := hh'h^{-1}h'^{-1}$ of $h, h' \in \hat{G}$ is in $\ker \pi = i(N)$, and hence in the center of $\hat{G}$.

If we take a section $s : G \to \hat{G}$, we can define a function $c : G \times G \to N$ by

$$c(g, g') = i^{-1}([s(g), s(g')]). \tag{A.23}$$

If we take another section $s' : G \to \hat{G}$, then we have $[s'(g), s'(g')] = [s(g), s(g')]$, which follows from the fact that $s'(g)s(g)^{-1}$ is in $\ker \pi = i(N)$ and hence in the center of $\hat{G}$. Therefore, the function c does not depend on the choice of the section. This c is called the *commutator map* associated with the central extension.

The properties of the commutator, $[h, h] = 1$ and $[h, h'] = [h', h]^{-1}$, respectively translates to the *alternating* property

$$c(g, g) = 1_N, \tag{A.24}$$

and the *antisymmetric* property

$$c(g, g') = c(g', g)^{-1}, \tag{A.25}$$

of the commutator map c.

From the fact that any commutator of $\hat{G}$ is in the center of $\hat{G}$, we also have the properties

$$[hh', h''] = [h, h''][h', h''], \tag{A.26}$$

$$[h, h'h''] = [h, h'][h, h''], \tag{A.27}$$

for any $h, h', h'' \in \hat{G}$. They translates to the *bilinearity*

$$c(gg', g'') = c(g, g'')c(g', g''), \tag{A.28}$$

$$c(g, g'g'') = c(g, g')c(g, g''), \tag{A.29}$$

of the commutator map c. Under the bilinearity (A.28, A.29), the antisymmetric property (A.25) follows from the alternating property (A.24).[45]

In terms of the cocycle ϵ of the section s, defined in (A.2), the commutator map is

$$c(g, g') = \epsilon(g, g')\epsilon(g', g)^{-1}. \tag{A.30}$$

This (A.30) holds even if we replace ϵ with another cocycle ϵ' cohomologous to ϵ, because such ϵ' is, as we have seen around (A.7), just a cocycle for another section. (Or, we can explicitly calculate $\epsilon'(g, g')\epsilon'(g', g)^{-1} = \epsilon(g, g')\epsilon(g', g)^{-1}$.) Therefore, (A.30) establishes a map

$$\flat_c : H^2(G, N) \to \left\{ \begin{array}{c} \text{alternating bilinear maps} \\ c : G \times G \to N \end{array} \right\} \tag{A.31}$$

[45] The converse ("antisymmetric $\Rightarrow$ alternating" under bilinearity) holds if N does not have order-2 elements.

for abelian groups G and N, where G acts on N trivially.

To obtain the inverse $\sharp_c$ of this map $\flat_c$, we further assume that G be a free abelian group of finite rank r, and take a $\mathbb{Z}$-basis $e_1, \ldots, e_r$ of G. For a given alternating $\mathbb{Z}$-bilinear map $c : G \times G \to N$, we define a $\mathbb{Z}$-bilinear map $\epsilon_c : G \times G \to N$ by the bilinear extension of

$$\epsilon_c(e_i, e_j) = \begin{cases} c(e_i, e_j) & (i > j), \\ 1_N & (i \leq j). \end{cases} \tag{A.32}$$

Since the action of G on N is trivial in the current situation, a $\mathbb{Z}$-bilinear map $G \times G \to N$ is automatically a 2-cocycle. Now, this cocycle ϵ_c satisfies (A.30) for the given c. Therefore, $c \mapsto \epsilon_c$ establishes a map

$$\sharp_c : \left\{ \begin{array}{c} \text{alternating } \mathbb{Z}\text{-bilinear maps} \\ c : G \times G \to N \end{array} \right\} \to H^2(G, N) \tag{A.33}$$

for a free abelian group G of finite rank and an abelian group N, and $\flat_c \circ \sharp_c$ is an identity map.

To see that $\sharp_c \circ \flat_c$ is also an identity map, we start from a given cocycle ϵ, consider the commutator map c_ϵ for ϵ as in (A.30), and show that the cocycle ϵ_{c_ϵ} constructed from this c_ϵ as in (A.32) is in the same cohomology class as the original cocycle ϵ. Let $\hat{G}_\epsilon$ be the central extension constructed from ϵ as in (A.10). Recall that the cocycle for the section $s : G \to \hat{G}_\epsilon \,; g \mapsto (1_N, g)$ is ϵ itself, and hence the commutator map associated with $\hat{G}_\epsilon$ is of course c_ϵ. Take another section $s' : G \to \hat{G}_\epsilon$ as follows: set $s'(e_1), \ldots, s'(e_r)$ to any element satisfying $\pi(s'(e_i)) = e_i$, say $s'(e_i) = s(e_i)$, and for a general element $e_1^{k^1} \cdots e_r^{k^r} \in G$, define

$$s'(e_1^{k^1} \cdots e_r^{k^r}) = s'(e_1)^{k^1} \cdots s'(e_r)^{k^r}. \tag{A.34}$$

This $s' : G \to \hat{G}_\epsilon$ is well-defined because G is a free abelian group, that is, torsion-free.[46] Note that this s' is not necessarily linear, or a homomorphism, because $s'(e_i)$'s are not necessarily commutative. Then the cocycle ϵ' for this section s' coincides with ϵ_{c_ϵ}, because

$$s'(e_1^{k^1} \cdots e_r^{k^r}) \cdot s'(e_1^{l^1} \cdots e_r^{l^r}) \tag{A.35}$$
$$= s'(e_1)^{k^1} \cdots s'(e_r)^{k^r} \cdot s'(e_1)^{l^1} \cdots s'(e_r)^{l^r} \tag{A.36}$$
$$= s'(e_1)^{k^1} \cdots s'(e_{r-1})^{k^{r-1}} \cdot s'(e_1)^{l^1} \cdots s'(e_r)^{k^r + l^r} i(\prod_{r>j} c_\epsilon(e_r, e_j)^{k^r l^j}) \tag{A.37}$$
$$= \cdots = s'(e_1)^{k^1 + l^1} \cdots s'(e_r)^{k^r + l^r} i(\prod_{i>j} c_\epsilon(e_i, e_j)^{k^i l^j}) \tag{A.38}$$
$$= s'(e_1^{k^1+l^1} \cdots e_r^{k^r+l^r}) i(\epsilon_{c_\epsilon}(e_1^{k^1} \cdots e_r^{k^r}, e_1^{l^1} \cdots e_r^{l^r})), \tag{A.39}$$

46 If G is a finitely generated abelian group, where torsion is allowed, then we cannot take a well-defined section s' in the form of (A.34) in general, and in particular, we cannot transform (A.38) into (A.39). In fact, there are inequivalent extensions of G with the same commutator map in this case; see Remark A.6 for example. If we further assume that N is a divisible group, then we can take a well-defined section s' in the form of (A.34) even if G has a torsion part, and therefore we can still establish the one-to-one correspondence with commutator maps. For example, [Tam00, Prop. 2.6] deals with the case of $N = k^\times$, which is the multiplicative group of a field k.

where in the second equation, we used the fact that the commutator map c_ϵ does not depend on the choice of the section s or s', as we saw above. Therefore, ϵ and ϵ_{c_ϵ} are the cocycles of the different sections s and s' of the same extension $\hat{G}_\epsilon$, and hence they differ only by the coboundary, as we have seen around (A.7).

We finally obtained the following theorem.

Theorem A.2. *For a free abelian group G of finite rank and an abelian group N, there exist one-to-one correspondences*

$$\{\textit{central extensions of } G \textit{ by } N\}\,/\textit{equivalence} \underset{\sharp}{\overset{\flat}{\rightleftarrows}} H^2(G,N) \tag{A.40}$$

$$\underset{\sharp_c}{\overset{\flat_c}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{alternating } \mathbb{Z}\textit{-bilinear maps} \\ c : G \times G \to N \end{array} \right\}, \tag{A.41}$$

where in $H^2(G,N)$, N is regarded as a G-module by the trivial G-action.

A.3 Central Extension of Free Abelian Group by $\mathbb{Z}_2$ and Quadratic Forms

A central extension of a free abelian group, say a lattice L, by $\mathbb{Z}_2$ plays a major role in discussions of a lattice VOA. In this Section A.3, following discussions around [FLM88, Prop. 5.3.4], we will see that we can induce central extensions of $L/2L$ by $\mathbb{Z}_2$ by specifying quadratic forms, which is particularly important when we construct the $\mathbb{Z}_2$-twisted sector in Section 5.3.1.

Let L be a free abelian group of finite rank r, and we regard it as an additive group, that is, the operation is denoted by $+$ and the identity element is 0. Let $\hat{L}$ be the central extension of L by $\mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1 \rangle$

$$1 \to \mathbb{Z}_2 \xrightarrow{i} \hat{L} \xrightarrow{\pi} L \to 0, \tag{A.42}$$

specified by a commutator map $c : L \times L \to \mathbb{Z}_2$. Since c is bilinear, c induces $\tilde{c} : L/2L \times L/2L \to \mathbb{Z}_2$. $\tilde{c}$ is also alternating and bilinear. $L/2L$ is isomorphic to $(\mathbb{F}_2)^r$ as an $\mathbb{F}_2$-linear vector space.

In general, for a given alternating bilinear form $\tilde{c} : (\mathbb{F}_2)^r \times (\mathbb{F}_2)^r \to \mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1 \rangle$, there is a map $\tilde{q} : (\mathbb{F}_2)^r \to \mathbb{Z}_2$ such that

$$\tilde{c}(x,y) = \tilde{q}(x+y)\tilde{q}(x)^{-1}\tilde{q}(y)^{-1}, \tag{A.43}$$

and the set of all such maps is in the form of $\{\tilde{q}\eta \mid \eta : (\mathbb{F}_2)^r \to \mathbb{Z}_2 \text{ linear}\}$ [FLM88, Remark 5.3.2].[47] Conversely, a map $\tilde{q} : (\mathbb{F}_2)^r \to \mathbb{Z}_2$ such that the map $\tilde{c} : (\mathbb{F}_2)^r \times (\mathbb{F}_2)^r \to \mathbb{Z}_2$ defined by (A.43) is bilinear is called a *quadratic form*, and $\tilde{c}$ is called the *associated form* of $\tilde{q}$. In such a situation, $\tilde{c}$ is obviously alternating (or equivalently, symmetric in $\mathbb{Z}_2$) by the definition (A.43).

[47] The proof of this statement is as follows. Take a $\mathbb{F}_2$-basis $e_1, \dots, e_r$ of $(\mathbb{F}_2)^r$, and set $\tilde{q}(e_i)$ $(i = 1, \dots, r)$ as arbitrary values of $\mathbb{Z}_2$. Define $\tilde{\epsilon} : (\mathbb{F}_2)^r \times (\mathbb{F}_2)^r \to \mathbb{Z}_2$ as the bilinear extension of (A.49) for $\tilde{c}$ and $\tilde{q}$ instead of c and q. Then $\tilde{q}(-) := \tilde{\epsilon}(-,-)$ satisfies (A.43). The degrees of freedom in choosing the values of $\tilde{q}(e_i)$ $(i = 1, \dots, r)$ correspond to the multiplication of linear maps $\eta : (\mathbb{F}_2)^r \to \mathbb{Z}_2$.

Remark A.3. More generally, a *quadratic form* on a $\mathbb{K}$-vector space V is a map $\tilde{q} : V \to \mathbb{K}$ such that

$$\tilde{q}(tv) = t^2 \tilde{q}(v) \quad (t \in \mathbb{K}, v \in V), \tag{A.44}$$

and its *associated form* or *polarization* $\tilde{b} : V \times V \to \mathbb{K}$ defined by

$$\tilde{b}(v, w) := \tilde{q}(v + w) - \tilde{q}(v) - \tilde{q}(w) \tag{A.45}$$

is bilinear.

If the characteristic of the field $\mathbb{K}$ is not 2, then the quadratic forms and the symmetric bilinear forms are in one-to-one correspondence under $\tilde{q} \mapsto \tilde{b}$; the converse $\tilde{b} \mapsto \tilde{q}$ is $\tilde{q}(v) = \frac{1}{2}\tilde{b}(v, v)$. In this case, the associated form is sometimes defined as

$$\tilde{b}'(v, w) := \frac{1}{2} \left(\tilde{q}(v + w) - \tilde{q}(v) - \tilde{q}(w) \right). \tag{A.46}$$

Note that if we represent a symmetric bilinear form $\tilde{b}'(v, w)$ as $v^T B w$ by a symmetric matrix B, then the corresponding quadratic form $\tilde{q}(v)$ is $v^T B v$.

However, if the characteristic of $\mathbb{K}$ is 2, then this one-to-one correspondence does not hold. In the special case $\mathbb{K} = \mathbb{F}_2$, the condition (A.44) is always satisfied, and the definition of the quadratic form reduces to the one we already defined before this Remark A.3. As mentioned above, there are multiple quadratic forms with the same associated form, and they are parametrized by linear maps $\eta : V \to \mathbb{F}_2$. One explanation of this special property of $\mathbb{F}_2$ is that any linear map η satisfies (A.44). Another explanation is that, over $\mathbb{F}_2$, a linear map $\eta : V \to \mathbb{F}_2$ can be represented as $\eta(v) = v^T \mathrm{diag}(\eta(e_1), \ldots, \eta(e_r)) v$ with an $\mathbb{F}_2$-basis $e_1, \ldots, e_r$ of V. Therefore, we can add arbitrary linear maps η to a quadratic form to get another quadratic form with the same associated form. The resulting quadratic form is $\tilde{q}(v) = v^T [\tilde{\epsilon}(e_i, e_j)]_{i,j} v$ in the language of footnote 47. *(Remark ends.)*

Getting back to the $\tilde{c} : L/2L \times L/2L \to \mathbb{Z}_2$ induced from the commutator map c of the central extension $\hat{L}$ (A.42), let us take a quadratic map $\tilde{q} : L/2L \to \mathbb{Z}_2$ with associated form $\tilde{c}$. Now, we will show that a lift of $2L$

$$K_{\tilde{q}} := \{ i(\tilde{q}(k + 2L)) \hat{k}^2 \mid \hat{k} \in \hat{L}, k = \pi(\hat{k}) \} \tag{A.47}$$

is in fact a subgroup of $\hat{L}$ contained in the center of $\hat{L}$.

Let $q : L \to \mathbb{Z}_2$ be the pullback $(L \to L/2L) \circ \tilde{q}$ of $\tilde{q}$. Obviously we have

$$c(k, k') = q(k + k') q(k)^{-1} q(k')^{-1}. \tag{A.48}$$

By taking a $\mathbb{Z}$-basis $e_1, \ldots, e_r$ of L, we can construct a cocycle $\epsilon : L \times L \to \mathbb{Z}_2$ whose commutator $\epsilon(k, k')\epsilon(k', k)^{-1}$ coincides with $c(k, k')$, by the bilinear extension of

$$\epsilon(e_i, e_j) = \begin{cases} c(e_i, e_j) & (i > j), \\ q(e_i) & (i = j), \\ 1 & (i < j). \end{cases} \tag{A.49}$$

(Note that this construction of a cocycle can produce a different one from (A.32) in general.)

Lemma A.4.

(1) $q((t_1+t_2)k) = q(t_1k)q(t_2k)$ *for any* $t_1, t_2 \in \mathbb{Z}$ *and* $k \in L$.

(2) $\epsilon(k,k) = q(k)$ *for any* $k \in L$.

Proof. (1) Since c is alternating and bilinear, $c(t_1k, t_2k) = 1$. Use it to (A.48).

(2) Let us write $k = \sum_i k^i e_i$. Using (1) and the bilinearity of c to the definition (A.49) of ϵ, we have

$$\epsilon(k,k) = \prod_{i>j} c(k^i e_i, k^j e_j) \prod_i q(k^i e_i). \tag{A.50}$$

Then, using (A.48) and the bilinearity of c repeatedly,

$$\begin{aligned}
\epsilon(k,k) &= \prod_{i\geq 3, i>j} c(k^i e_i, k^j e_j) \cdot c(k^2 e_2, k^1 e_1) q(k^1 e_1) q(k^2 e_2) \cdot \prod_{i \geq 3} q(k^i e_i) && \text{(A.51)} \\
&= \prod_{i\geq 3, i>j} c(k^i e_i, k^j e_j) \cdot q(k^1 e_1 + k^2 e_2) \cdot \prod_{i \geq 3} q(k^i e_i) && \text{(A.52)} \\
&= \prod_{i\geq 4, i>j} c(k^i e_i, k^j e_j) \cdot c(k^3 e_3, k^1 e_1 + k_2 e^2) q(k^1 e_1 + k^2 e_2) q(k^3 e_3) \cdot \prod_{i \geq 4} q(k^i e_i) && \text{(A.53)} \\
&= \prod_{i\geq 4, i>j} c(k^i e_i, k^j e_j) \cdot q(k^1 e_1 + k^2 e_2 + k^3 e_3) \cdot \prod_{i \geq 4} q(k^i e_i) && \text{(A.54)} \\
&\vdots && \text{(A.55)} \\
&= q(k^1 e_1 + \cdots + k^r e_r) && \text{(A.56)} \\
&= q(k). && \text{(A.57)}
\end{aligned}$$

□

We take a section $s : L \to \hat{L}$ of (A.42) such that its cocycle is ϵ. Then, for any $\kappa^m s(k) \in \hat{L}$ (we omit $i : \mathbb{Z}_2 \to \hat{L}$),

$$\begin{aligned}
q(\pi(\kappa^m s(k)))(\kappa^m s(k))^2 &= q(k)s(k)s(k) && \text{(A.58)} \\
&= q(k)\epsilon(k,k)s(2k) && \text{(A.59)} \\
&= s(2k), && \text{(A.60)}
\end{aligned}$$

where we used Lemma A.4 (2) in the last equation. As a result, $K_{\tilde{q}}$ defined in (A.47) can be written as

$$K_{\tilde{q}} = \{s(2k) \mid k \in L\}. \tag{A.61}$$

In addition, since ϵ is bilinear, $\epsilon(2k, k') = \epsilon(k', 2k) = 1$. This shows that $K_{\tilde{q}}$ is a subgroup of $\hat{L}$, and it is in the center of $\hat{L}$.

Now, we have a central extension

$$1 \to \mathbb{Z}_2 \xrightarrow{\iota} \hat{L}/K_{\tilde{q}} \to L/2L \to 0, \tag{A.62}$$

such that for any $\kappa^m s(k) \in \hat{L}$,

$$(\kappa^m s(k) K_{\tilde{q}})^2 = \epsilon(k,k) s(2k) K_{\tilde{q}} = q(k) K_{\tilde{q}} = \iota(\tilde{q}(k+2L)), \tag{A.63}$$

where we used Lemma A.4 (2) in the second equation.

The following Theorem A.5 is known. From this viewpoint of Theorem A.5 (2), the equation (A.63) shows that the central extension (A.62) of $L/2L$ by $\mathbb{Z}_2$ is the one specified by the quadratic form $\tilde{q}$.

Theorem A.5 ([FLM88, Prop. 5.3.3])**.**

(1) For a central extension

$$1 \to \mathbb{Z}_2 \xrightarrow{i} \hat{E} \xrightarrow{\pi} (\mathbb{F}_2)^r \to 0, \tag{A.64}$$

whose commutator map is $\tilde{c} : (\mathbb{F}_2)^r \times (\mathbb{F}_2)^r \to \mathbb{Z}_2$*, if we define a map* $\tilde{q} : (\mathbb{F}_2)^r \to \mathbb{Z}_2$ *by*

$$a^2 = i(\tilde{q}(\pi(a))) \quad \textit{for} \quad a \in \hat{E}, \tag{A.65}$$

then $\tilde{q}$ *is a quadratic form with its associated form* $\tilde{c}$*.*

(2) The association of a quadratic form to a central extension in (1) establishes a one-to-one correspondence

$$\{\textit{central extensions of } (\mathbb{F}_2)^r \textit{ by } \mathbb{Z}_2\} /\textit{equivalence} \leftrightarrow \left\{ \begin{array}{c} \textit{quadratic forms} \\ \tilde{q} : (\mathbb{F}_2)^r \to \mathbb{Z}_2 \end{array} \right\}. \tag{A.66}$$

Remark A.6. As we have mentioned below (A.43), there are multiple quadratic forms with the same associated form. Therefore, Theorem A.5 is an example where the one-to-one correspondence between equivalence classes of central extensions and commutator maps like Theorem A.2 does not hold. (See also footnote 46.) *(Remark ends.)*

To summarize the above discussions, we obtained the following theorem.

Theorem A.7 ([FLM88, Prop. 5.3.4])**.** *Let* $\hat{L}$ *be the central extension of a free abelian group* L *by* $\mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1 \rangle$

$$1 \to \mathbb{Z}_2 \xrightarrow{i} \hat{L} \xrightarrow{\pi} L \to 0, \tag{A.67}$$

specified by a commutator map $c : L \times L \to \mathbb{Z}_2$*.* c *induces the alternating bilinear map* $\tilde{c} : L/2L \times L/2L \to \mathbb{Z}_2$*. For any quadratic form* $\tilde{q} : L/2L \to \mathbb{Z}_2$ *with associated form* $\tilde{c}$*,*

$$K_{\tilde{q}} := \{ i(\tilde{q}(k+2L))\hat{k}^2 \mid \hat{k} \in \hat{L}, k = \pi(\hat{k}) \} \tag{A.68}$$

is a central subgroup of $\hat{L}$*, and*

$$1 \to \mathbb{Z}_2 \to \hat{L}/K \to L/2L \to 0 \tag{A.69}$$

is a central extension with quadratic form $\tilde{q}$*.*

A.4 A Theorem on Automorphism Group of Central Extension

This Section A.4 is a review of [FLM88, Prop. 5.4.1].

Let $1 \to N \xrightarrow{i} \hat{G} \xrightarrow{\pi} G \to 1$ be a central extension of a free abelian group G of finite rank by an abelian group N, and $c : G \times G \to N$ be the associated commutator map.

We first define some necessary objects. We define a subgroup $\mathrm{Aut}(\hat{G}, N)$ of the automorphism group $\mathrm{Aut}(\hat{G})$ as

$$\mathrm{Aut}(\hat{G}, N) := \{\chi \in \mathrm{Aut}(\hat{G}) \mid \chi(i(n)) = i(n) \text{ for any } n \in N\}. \tag{A.70}$$

Since N is abelian, $\mathrm{Hom}(G, N)$ has a group structure under the multiplication of functions: for $\eta, \eta' \in \mathrm{Hom}(G, N)$, $(\eta \cdot \eta')(g) = \eta(g)\eta'(g)$. Now we can define a group homomorphism $I : \mathrm{Hom}(G, N) \to \mathrm{Aut}(\hat{G}, N)$ which maps $\eta \in \mathrm{Hom}(G, N)$ to

$$I(\eta) : \hat{G} \to \hat{G} \tag{A.71}$$
$$h \mapsto i(\eta(\pi(h)))h. \tag{A.72}$$

In fact, $I(\eta)$ is an element of $\mathrm{Aut}(\hat{G}, N)$, and I satisfies $I(\eta \cdot \eta') = I(\eta) \circ I(\eta')$.

We also define a subgroup $\mathrm{Aut}(G, c)$ of $\mathrm{Aut}(G)$ as

$$\mathrm{Aut}(G, c) := \{\phi \in \mathrm{Aut}(G) \mid c(\phi(g), \phi(g')) = c(g, g') \text{ for any } g, g' \in G\}, \tag{A.73}$$

and a group homomorphism $\Pi : \mathrm{Aut}(\hat{G}, N) \to \mathrm{Aut}(G, c)$ which maps $\chi \in \mathrm{Aut}(\hat{G}, N)$ to

$$\Pi(\chi) : G \to G \tag{A.74}$$
$$g \mapsto \pi(\chi(\hat{g})), \tag{A.75}$$

where $\hat{g}$ is an element of $\hat{G}$ such that $\pi(\hat{g}) = g$. Here, $\Pi(\chi)(g)$ does not depend on the choice of $\hat{g}$, because such another $\hat{g}'$ is an element of $\hat{g}i(N)$, and hence

$$\pi(\chi(\hat{g}')) \in \pi(\chi(\hat{g}i(N))) = \pi(\chi(\hat{g})i(N)) = \{\pi(\chi(\hat{g}))\}. \tag{A.76}$$

To see that $\Pi(\chi)$ is in fact an element of $\mathrm{Aut}(G, c)$, it suffices to calculate

$$c(\Pi(\chi)(g), \Pi(\chi)(g')) = i^{-1}([\widehat{\pi(\chi(\hat{g}))}, \widehat{\pi(\chi(\widehat{g'}))}]) \tag{A.77}$$
$$= i^{-1}([\chi(\hat{g}), \chi(\widehat{g'})]) = i^{-1}(\chi([\hat{g}, \widehat{g'}])) = i^{-1}([\hat{g}, \widehat{g'}]) = c(g, g'), \tag{A.78}$$

where we used the definition (A.23) of the commutator map c in the form of $c(-,-) = i^{-1}([\hat{-}, \hat{-}])$; recall that it does not depend on the choice of $\hat{-}$. Lastly, it is easy to see that Π is in fact a group homomorphism: $\Pi(\chi \circ \chi') = \Pi(\chi) \circ \Pi(\chi')$.

Now, here is the theorem.

Theorem A.8 ([FLM88, Prop. 5.4.1]). *The following sequence is exact.*

$$1 \to \mathrm{Hom}(G, N) \xrightarrow{I} \mathrm{Aut}(\hat{G}, N) \xrightarrow{\Pi} \mathrm{Aut}(G, c) \to 1. \tag{A.79}$$

Proof. $\ker I = 1_N$ (constant function in $\mathrm{Hom}(G, N)$) and $\mathrm{im}\, I \subset \ker \Pi$ are easy.

$\ker \Pi \subset \mathrm{im}\, I$ can be checked as follows. Let $\chi \in \mathrm{Aut}(\hat{G}, N)$ satisfy $\Pi(\chi) = \mathrm{id}_G$. Since $\pi(\chi(\hat{g})) = g$ for any $g \in G$, we have $\pi(\chi(h)) = \pi(h)$ for any $h \in \hat{G}$, and hence the difference $\chi(h)h^{-1}$ of $\chi(h)$ and h is in $i(N)$. Furthermore, it only depends on $\pi(h)$, because if we take another $h' \in \hat{G}$ such that $\pi(h') = \pi(h)$, then $h' \in hi(N)$ and hence $\chi(h')(h')^{-1} = \chi(h)h^{-1}$. Therefore, there exists a map $\eta : G \to N$ such that $\chi(h)h^{-1} = i(\eta(\pi(h)))$. The linearity of η follows from, for any $h, h' \in \hat{G}$,

$$\chi(hh')(hh')^{-1} = \chi(h)\chi(h')(h')^{-1}h^{-1} = \chi(h)h^{-1}\chi(h')(h')^{-1}, \tag{A.80}$$

where the last equation follows from $\chi(h')(h')^{-1} \in i(N) \subset \mathrm{Center}(\hat{G})$. As a result, $\eta \in \mathrm{Hom}(G, N)$ and $I(\eta) = \chi$.

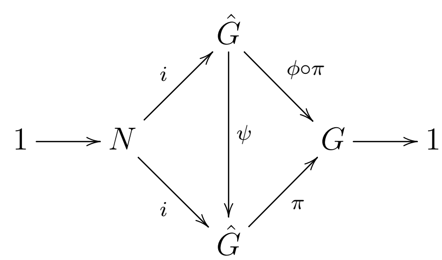
The surjectivity of Π can be shown as follows. For $\phi \in \mathrm{Aut}(G, c)$, consider a new central extension

$$1 \to N \xrightarrow{i} \hat{G} \xrightarrow{\phi \circ \pi} G \to 1. \tag{A.81}$$

If $s : G \to \hat{G}$ is a section of the original central extension, i.e. $\pi \circ s = \mathrm{id}_G$, then $s \circ \phi^{-1}$ is a section of (A.81). Then, the commutator map c_ϕ associated with the new central extension coincides with the original commutator map c, because

$$c_\phi(g, g') = c(\phi^{-1}(g), \phi^{-1}(g)) = c(g, g'). \tag{A.82}$$

Therefore, by Theorem A.2, there exists $\psi \in \mathrm{Aut}(\hat{G})$ such that the diagram



(A.83)

commutes. It is obvious that $\psi \in \mathrm{Aut}(\hat{G}, N)$ from the diagram, and $\Pi(\psi) = \phi$ because

$$\Pi(\psi)(g) = \pi(\psi(s(g))) = \phi(\pi(s(g))) = \phi(g). \tag{A.84}$$

□

B Notations of Theta Functions

In this Section B, we review the notations of several theta functions and the Dedekind eta function, and their modular transformations. We mainly follow [Pol07, Eqs. (7.2.31)–(7.2.44)], but $z_{\text{Polchinski}} = e^{2\pi\sqrt{-1}\nu_{\text{Polchinski}}}$ is $y = e^{2\pi\sqrt{-1}z}$ here, and the order of the arguments is (ν, τ) in [Pol07] but (τ, z) here, following [ES15, Appendix]. As always, $q = e^{2\pi\sqrt{-1}\tau}$.

The (basic) Jacobi theta function

We first define the *(basic) Jacobi theta function* as

$$\vartheta(\tau, z) := \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n = \prod_{m=1}^{\infty} (1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}). \tag{B.1}$$

The second equation is called the Jacobi triple product identity.

Its modular transformations are

$$\vartheta(\tau+1, z) = \vartheta(\tau, z+\frac{1}{2}), \tag{B.2}$$

$$\vartheta(-\frac{1}{\tau}, \frac{z}{\tau}) = (-\sqrt{-1}\tau)^{1/2} e^{\pi\sqrt{-1}z^2/\tau} \vartheta(\tau, z). \tag{B.3}$$

The modular T transformation (B.2) is obvious. The modular S transformation (B.3) can be shown as follows, which is the so-called Poisson summation formula. Using the Fourier expansion of the periodic delta function

$$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{k=-\infty}^{\infty} e^{2\pi\sqrt{-1}kx}, \tag{B.4}$$

we can calculate that

$$\vartheta(-\frac{1}{\tau}, \frac{z}{\tau}) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dx\; e^{-\frac{\pi\sqrt{-1}}{\tau}x^2 + 2\pi\sqrt{-1}(\frac{z}{\tau}+k)x} \tag{B.5}$$

$$= \sum_{k=-\infty}^{\infty} e^{\frac{\pi\sqrt{-1}}{\tau}(z+\tau k)^2} \int_{-\infty}^{\infty} dx\; e^{-\frac{\pi\sqrt{-1}}{\tau}(x-(z+\tau k))^2} \tag{B.6}$$

$$= \sum_{k=-\infty}^{\infty} e^{2\pi\sqrt{-1}(\tau\frac{k^2}{2} + zk + \frac{z^2}{2\tau})} \cdot (-\sqrt{-1}\tau)^{1/2} \tag{B.7}$$

$$= (-\sqrt{-1}\tau)^{1/2} e^{\pi\sqrt{-1}z^2/\tau} \vartheta(\tau, z). \tag{B.8}$$

In the third equation, we used the Gauss integral

$$\int_{-\infty}^{\infty} dx\; e^{-a(x-b)^2} = \sqrt{\frac{\pi}{a}} \quad (\mathrm{Re}(a) > 0,\; b \in \mathbb{C}). \tag{B.9}$$

The theta function with characteristics

The *theta function with characteristics* is defined as

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](\tau,z) := e^{2\pi\sqrt{-1}ab} q^{\frac{a^2}{2}} y^a \vartheta(\tau, z+a\tau+b) \tag{B.10}$$

$$= \sum_{n=-\infty}^{\infty} e^{2\pi\sqrt{-1}(n+a)b} q^{\frac{1}{2}(n+a)^2} y^{n+a} \tag{B.11}$$

$$= e^{2\pi\sqrt{-1}ab} q^{\frac{a^2}{2}} y^a \prod_{m=1}^{\infty} (1-q^m)(1+e^{2\pi\sqrt{-1}b} y q^{m-1/2+a})(1+e^{-2\pi\sqrt{-1}b} y^{-1} q^{m-1/2-a}). \tag{B.12}$$

Its modular transformations follow from (B.2, B.3) as

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](\tau+1,z) = e^{-\pi\sqrt{-1}(a^2+a)} \theta\left[\begin{array}{c} a \\ a+b+\frac{1}{2} \end{array}\right](\tau,z), \tag{B.13}$$

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](-\frac{1}{\tau},\frac{z}{\tau}) = (-\sqrt{-1}\tau)^{1/2} e^{\pi\sqrt{-1}z^2/\tau} e^{2\pi\sqrt{-1}ab} \theta\left[\begin{array}{c} b \\ -a \end{array}\right](\tau,z). \tag{B.14}$$

The following equations, which follow from (B.10) and (B.11), are also useful.

$$\theta\left[\begin{array}{c} a+l \\ b+m \end{array}\right](\tau,z) = e^{2\pi\sqrt{-1}am} \theta\left[\begin{array}{c} a \\ b \end{array}\right](\tau,z) \quad (l,m \in \mathbb{Z}), \tag{B.15}$$

$$\theta\left[\begin{array}{c} a+a' \\ b+b' \end{array}\right](\tau,z) = e^{2\pi\sqrt{-1}a'(b+b')} q^{\frac{(a')^2}{2}} y^{a'} \theta\left[\begin{array}{c} a \\ b \end{array}\right](\tau, z+a'\tau+b'), \tag{B.16}$$

$$\theta\left[\begin{array}{c} -a \\ -b \end{array}\right](\tau,z) = \theta\left[\begin{array}{c} a \\ b \end{array}\right](\tau,-z). \tag{B.17}$$

The elliptic theta functions

The *elliptic theta functions*, also often called the *Jacobi theta functions*, are defined as

$$\theta_3(\tau,z) := \theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](\tau,z) = \vartheta(\tau,z) = \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n, \tag{B.18}$$

$$\theta_4(\tau,z) := \theta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right](\tau,z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n, \tag{B.19}$$

$$\theta_2(\tau,z) := \theta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](\tau,z) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} y^{n+\frac{1}{2}}, \tag{B.20}$$

$$\theta_1(\tau,z) := -\theta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](\tau,z) = \sqrt{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}}. \tag{B.21}$$

We also write $\theta(\tau)$ for $\theta(\tau,0)$. Their product representations are, from (B.12),

$$\theta_3(\tau,z) = \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}), \tag{B.22}$$

$$\theta_4(\tau,z) = \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}), \tag{B.23}$$

$$\theta_2(\tau,z) = q^{1/8}y^{1/2}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^{m-1}) \tag{B.24}$$

$$= q^{1/8}(2\cos\pi z)\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m), \tag{B.25}$$

$$\theta_1(\tau,z) = -\sqrt{-1}q^{1/8}y^{1/2}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^{m-1}) \tag{B.26}$$

$$= q^{1/8}(2\sin\pi z)\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m). \tag{B.27}$$

Their modular transformations follow from (B.13, B.14) as

$$\theta_3(\tau+1,z) = \theta_4(\tau,z), \tag{B.28}$$

$$\theta_4(\tau+1,z) = \theta_3(\tau,z), \tag{B.29}$$

$$\theta_2(\tau+1,z) = e^{\frac{1}{4}\pi\sqrt{-1}}\theta_2(\tau,z), \tag{B.30}$$

$$\theta_1(\tau+1,z) = e^{\frac{1}{4}\pi\sqrt{-1}}\theta_1(\tau,z), \tag{B.31}$$

and

$$\theta_3(-\frac{1}{\tau},\frac{z}{\tau}) = (-\sqrt{-1}\tau)^{1/2}e^{\pi\sqrt{-1}z^2/\tau}\theta_3(\tau,z), \tag{B.32}$$

$$\theta_4(-\frac{1}{\tau},\frac{z}{\tau}) = (-\sqrt{-1}\tau)^{1/2}e^{\pi\sqrt{-1}z^2/\tau}\theta_2(\tau,z), \tag{B.33}$$

$$\theta_2(-\frac{1}{\tau},\frac{z}{\tau}) = (-\sqrt{-1}\tau)^{1/2}e^{\pi\sqrt{-1}z^2/\tau}\theta_4(\tau,z), \tag{B.34}$$

$$\theta_1(-\frac{1}{\tau},\frac{z}{\tau}) = -\sqrt{-1}(-\sqrt{-1}\tau)^{1/2}e^{\pi\sqrt{-1}z^2/\tau}\theta_1(\tau,z). \tag{B.35}$$

Jacobi's identity is

$$\theta_3(\tau,z)^4 - \theta_4(\tau,z)^4 - \theta_2(\tau,z)^4 + \theta_1(\tau,z)^4 = 0. \tag{B.36}$$

Note that

$$\theta_1(\tau,0) = 0. \tag{B.37}$$

The Dedekind eta function

The *Dedekind eta function* is defined as

$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3}{2}(n-\frac{1}{6})^2}. \tag{B.38}$$

The second equation can be obtained by replacing q and y in $\theta_3(\tau, z)$ with q^3 and $-q^{-\frac{1}{2}}$, respectively. The modular transformations are

$$\eta(\tau + 1) = e^{\pi\sqrt{-1}/12}\eta(\tau), \tag{B.39}$$

$$\eta(-\frac{1}{\tau}) = (-\sqrt{-1}\tau)^{1/2}\eta(\tau). \tag{B.40}$$

The modular T transformation (B.39) is obvious. To see the modular S transformation (B.40), we first observe that, from (B.27),

$$2\pi\eta(\tau)^3 = \partial_z\theta_1(\tau, z)|_{z=0}. \tag{B.41}$$

Then we have

$$2\pi\eta(-\frac{1}{\tau})^3 = \partial_z\theta_1(-\frac{1}{\tau}, z)|_{z=0} \tag{B.42}$$

$$= \tau\partial_z\theta_1(-\frac{1}{\tau}, \frac{z}{\tau})|_{z=0} \tag{B.43}$$

$$= \tau(-\sqrt{-1})(-\sqrt{-1}\tau)^{1/2}\partial_z\theta_1(\tau, z)|_{z=0} \tag{B.44}$$

$$= (-\sqrt{-1}\tau)^{3/2}2\pi\eta(\tau)^3. \tag{B.45}$$

C Cocycle Factor

To realize the appropriate commutation relations of the vertex operators $V_k(z) \propto :e^{\sqrt{-1}k\cdot X(z)}:$ ($k \in L$) of a lattice VOA V_L, in accordance with whether $V_k(z)$ is bosonic or fermionic, we have to introduce a correction factor $c_k(p)$ to modify the commutation relations of $:e^{\sqrt{-1}k\cdot X(z)}:$'s. As a result, an additional factor $\varepsilon(k,k')$ called a cocycle factor appears in the OPEs of the vertex operators. In this Appendix C, we will review this story in the language of physics. A good reference for cocycle factors is [GSW12, §§6.4.4-6.4.5], but it deals with only even lattices. The cases including odd lattices are discussed in for example [GO85, Appendix], [GTVW13, Appendix A].

As already mentioned, let L be an integral Euclidean lattice of rank n in these notes. We would like to construct the vertex operators $V_k(z)$ satisfying the commutation relation

$$V_k(z_1)\cdot V_{k'}(z_2) = (-1)^{|k|^2|k'|^2} V_{k'}(z_2)\cdot V_k(z_1), \tag{C.1}$$

for any $k,k' \in L$. The conformal weight of $V_k(z)$ is $\frac{1}{2}|k|^2$, so it is bosonic when k is even and fermionic when k is odd, which accounts for the factor $(-1)^{|k|^2|k'|^2}$ of (C.1).

Recalling that the OPE of operators $:e^{\sqrt{-1}k\cdot X(z)}:$ is

$$:e^{\sqrt{-1}k\cdot X(z_1)}:\cdot :e^{\sqrt{-1}k'\cdot X(z_2)}: = (z_1-z_2)^{k\cdot k'} :e^{\sqrt{-1}k\cdot X(z_1)} e^{\sqrt{-1}k'\cdot X(z_2)}:, \tag{C.2}$$

we can observe

$$:e^{\sqrt{-1}k\cdot X(z_1)}:\cdot :e^{\sqrt{-1}k'\cdot X(z_2)}: = (-1)^{k\cdot k'} :e^{\sqrt{-1}k'\cdot X(z_2)}:\cdot :e^{\sqrt{-1}k\cdot X(z_1)}: \sim (z_1-z_2)^{k\cdot k'} :e^{\sqrt{-1}(k+k')\cdot X(z_2)}:, \tag{C.3}$$

where $O((z_1-z_2)^{k\cdot k'+1})$ terms are dropped. This (C.3) differs from the desired commutation relation (C.1) only by the sign. To modify it, let us introduce the correction factors $c_k(p)$ as

$$V_k(z) = :e^{\sqrt{-1}k\cdot X(z)}: c_k(p), \tag{C.4}$$

where $c_k(p)$ is an operator in the form of a function of the momentum operators $p^i = \alpha_0^i$ ($i = 1,\dots,n$). Then (C.1) translates to the condition on this correction factor $c_k(p)$ as

$$c_k(p+k')c_{k'}(p) = (-1)^{k\cdot k' + |k|^2|k'|^2} c_{k'}(p+k)c_k(p), \tag{C.5}$$

where we used $c_k(p)\cdot :e^{\sqrt{-1}k'\cdot X(z)}: = :e^{\sqrt{-1}k'\cdot X(z)}:\cdot c_k(p+k')$, because $[p^i, :e^{\sqrt{-1}k'\cdot X(z)}:] = (k')^i :e^{\sqrt{-1}k'\cdot X(z)}:$.

In addition, if we assume that the OPE of $V_k(z_1)\cdot V_{k'}(z_2)$ contains $(z_1-z_2)^{k\cdot k'}V_{k+k'}(z_2)$ as expected from (C.3), then we need $c_k(p+k')c_{k'}(p) = c_{k+k'}(p)$. However, this immediately turns out to be impossible in general, because (C.5) states that $c_k(p+k')c_{k'}(p)$ and $c_{k'}(p+k)c_k(p)$, both of which are supposed to become $c_{k+k'}(p)$, can differ by sign. So, we introduce a factor $\varepsilon : L\times L \to \{\pm 1\}$ so that

$$c_k(p+k')c_{k'}(p) = (-1)^{k\cdot k'+|k|^2|k'|^2} c_{k'}(p+k)c_k(p) = \varepsilon(k,k')c_{k+k'}(p). \tag{C.6}$$

This allows the vertex operators $V_k(z)$ to satisfy the OPE

$$V_k(z_1) \cdot V_{k'}(z_2) = (-1)^{|k|^2|k'|^2} V_{k'}(z_2) \cdot V_k(z_1) \sim \varepsilon(k, k')(z_1 - z_2)^{k \cdot k'} V_{k+k'}(z_2), \tag{C.7}$$

where $O((z_1 - z_2)^{k \cdot k'+1})$ terms are dropped. It easily follows from (C.6) that ε must satisfy

$$\varepsilon(k, k') = (-1)^{k \cdot k' + |k|^2|k'|^2} \varepsilon(k', k). \tag{C.8}$$

Furthermore, we impose the associativity on $c_k(p)$ as

$$(c_k(p + k' + k'')c_{k'}(p + k''))c_{k''}(p) = c_k(p + k' + k'')(c_{k'}(p + k'')c_{k''}(p)). \tag{C.9}$$

Then the factor ε should satisfy the condition

$$\varepsilon(k, k')\varepsilon(k + k', k'') = \varepsilon(k, k' + k'')\varepsilon(k', k''). \tag{C.10}$$

This means that the factor $\varepsilon : L \times L \to \{\pm 1\}$ is a 2-cocycle in the language of group cohomology (where $\{\pm 1\}$ is regarded as an L-module by the trivial L-action), and hence ε is called a *cocycle factor*. A 2-cocycle satisfying (C.8) is unique up to coboundary; see Theorem A.2. In the language of Section A.2, ε is a 2-cocycle for the commutator map $c(k, k') = (-1)^{k \cdot k' + |k|^2|k'|^2}$.

To construct $c_k(p)$ satisfying (C.6) and (C.9), we choose a $\mathbb{Z}$-basis $\{e_i\}_{i=1,\dots,n}$ of L and introduce a bilinear non-commutative product $* : L \times L \to \mathbb{Z}$ on L by

$$k * k' := \sum_{i>j} k^i k'^j (e_i \cdot e_j + |e_i|^2|e_j|^2) + \sum_{i=1}^{n} k^i k'^i \frac{1}{2}(|e_i|^2 + |e_i|^4), \tag{C.11}$$

where $|e_i|^4 := (|e_i|^2)^2$, for $k = \sum_i k^i e_i$ and $k' = \sum_j k'^j e_j$. This product $*$ depends on the choice of the basis $\{e_i\}_i$. In the language of Section A.3, this definition (C.11) follows the construction (A.49) of a cocycle from the commutator map $c(k, k') = (-1)^{k \cdot k' + |k|^2|k'|^2}$ and the quadratic form $q(k) = (-1)^{\frac{1}{2}(|k|^2 + |k|^4)}$. This construction also appears in [Kac98, Remark 5.5a].

We can now construct $c_k(p)$ satisfying (C.6) and (C.9) as

$$c_k(p) = (-1)^{k * p}, \tag{C.12}$$

and then ε is given by

$$\varepsilon(k, k') = (-1)^{k * k'}, \tag{C.13}$$

which of course satisfies (C.8) and (C.10).

Proof that $c_k(p)$ in (C.12) satisfies (C.6) and (C.9). For the first equation of (C.6), it suffices to check

$$(-1)^{k * k'} = (-1)^{k \cdot k' + |k|^2|k'|^2} (-1)^{k' * k}. \tag{C.14}$$

This is already discussed around (A.49), but we can check it explicitly as follows:

$$(-1)^{k*k'+k'*k} = (-1)^{\sum_{i\neq j}(k^i k'^j e_i\cdot e_j + |k^i e_i|^2|k'^j e_j|^2)+\sum_i (k^i k'^i e_i\cdot e_i + |k^i e_i|^2|k'^i e_i|^2)} \tag{C.15}$$

$$= (-1)^{k\cdot k' + |k|^2|k'|^2}, \tag{C.16}$$

where the first equation follows from $k^i \equiv (k^i)^2 \bmod 2$, and the second equation follows from $\sum_i |k^i e_i|^2 \equiv |k|^2 \bmod 2$.

The second equation of (C.6), and (C.9), are obvious. □

Remark C.1. The 2-cocycle satisfying (C.8) is unique only up to coboundary. In particular, the bilinear non-commutative product we can use to construct $c_k(p)$ is not unique. For example, even if we define

$$k *_c k' := \sum_{i>j} k^i k'^j (e_i \cdot e_j + |e_i|^2 |e_j|^2), \tag{C.17}$$

and construct $c_k(p)$ as $(-1)^{k *_c p}$, it satisfies (C.6) and (C.9). In the language of Section A.2, this definition (C.17) follows the construction (A.32) of a cocycle from the commutator map $c(k,k') = (-1)^{k\cdot k' + |k|^2|k'|^2}$, without considering to specify a quadratic form.

If the lattice L is even, or if the lattice L is odd and we choose the basis so that e_1 is odd and $e_2, \ldots, e_n$ are even,[48] then this product reduces modulo 2 to

$$k *_c k' \equiv \sum_{i>j} k^i k'^j e_i \cdot e_j \mod 2. \tag{C.18}$$

This is the form adopted in [GSW12, §6.4.5].

This product $*_c$ (C.17) often suffices, but does not work well when we consider the reflection $\mathbb{Z}_2$ symmetry discussed in Section 5.3. Suppose we used this product $*_c$ to construct the cocycle factor ε, which will play the role of the 2-cocycle of the section $k \mapsto e^k$ of the extension $\hat{L}$ (5.36). Then $\varepsilon(k,k) = (-1)^{\frac{1}{2}|k|^2}$ in (5.74) no longer holds, because the information of the quadratic form $q(k) = (-1)^{\frac{1}{2}|k|^2}$ (which coincides with $q(k) = (-1)^{\frac{1}{2}(|k|^2+|k|^4)}$ for an even lattice) is not taken into account (See Lemma A.4 (2)). As a result, the equation (5.74) would be modified by sign, so if we follow the definition of θ_0 in [FLM88, Eq. (6.4.13)] as $\theta_0(e^k) = (-1)^{\frac{1}{2}|k|^2}(e^k)^{-1}$, then $\theta_0(e^k)$ would be e^{-k} only up to sign, and the reflection $\mathbb{Z}_2$ symmetry $|k\rangle \mapsto |-k\rangle$ would need to be modified by some extra signs $|k\rangle \mapsto \pm|-k\rangle$. To avoid this complication, we have adopted the product $*$ (C.11) in these notes. *(Remark ends.)*

Remark C.2. The product $*$ (C.11) is a little bit cumbersome. If we can take a $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of L such that

- if L is even,[49] $|e_i|^2 \in 4\mathbb{Z}$ for any $i = 1, \ldots, n$,

[48] This is always possible because (odd vector) − (odd vector) = (even vector).

[49] The existence of a $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of a lattice L satisfying $|e_i|^2 \in 4\mathbb{Z}$ for any $i = 1, \ldots, n$ does not imply that $|k|^2 \in 4\mathbb{Z}$ for any $k \in L$. In fact, we can take such a basis of the Leech lattice, as we can see from the symmetric bilinear form (2.49), but the Leech lattice has vectors whose squared lengths are not multiples of 4.

- if L is odd, $e_1 \in 4\mathbb{Z}+3$ and $|e_i|^2 \in 4\mathbb{Z}$ for any $i = 2, \ldots, n$,

then (C.11) modulo 2 coincides with the product (C.18)

$$k * k' \equiv \sum_{i>j} k^i k'^j e_i \cdot e_j \mod 2. \tag{C.19}$$

In the analysis in Sections 4.2 and 4.3 of [Oka24], the author used the bases satisfying these conditions, and this product (C.19) was used in the Python code there. *(Remark ends.)*

D Lift of $\mathrm{SO}(n)$ To $\mathrm{Spin}(n)$ and Its Action on Spinors

This Section D is a review of the spin group $\mathrm{Spin}(n)$ and the lift of elements of $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$. The spin group $\mathrm{Spin}(n)$ plays an important role in understanding the structures of the theory $A(\mathfrak{a})$ of free fermions and Duncan's module $V^{f\natural}$, which are constructed as the Clifford algebra modules in Section 6. We will review its definition and fundamental property as a double cover of $\mathrm{SO}(n)$ in Section D.1, and describe the explicit form of a lift of an element of $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$ in Section D.2.

D.1 Definitions of Pin Group and Spin Group

In this Section D.1, we review the definitions of pin group and spin group, and their fundamental properties as (double) covers of $\mathrm{O}(V)$ and $\mathrm{SO}(V)$, respectively. We will follow [LM89, Ch. I §2], so the details of the facts cited here can be found there.

To keep generality, let V be a vector space over a field $\mathbb{K}$ with a symmetric bilinear form $\langle -, - \rangle$. In addition, we assume that the definition of the Clifford algebra of V is

$$\mathrm{Cliff}(V) = T(V)/u \otimes u \sim \epsilon \langle u, u \rangle \quad (u \in V), \tag{D.1}$$

where ϵ is either $+1$ or -1 (cf. footnote 31). We will omit $\otimes$. The relation $u^2 = \epsilon\langle u, u\rangle$ is equivalent to

$$uv + vu = 2\epsilon \langle u, v \rangle \quad (u, v \in V). \tag{D.2}$$

The map $-\mathrm{id}_V : V \to V$ induces the parity involution $\theta : \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$, which defines the parity decomposition $\mathrm{Cliff}(V) = \mathrm{Cliff}(V)^0 \oplus \mathrm{Cliff}(V)^1$.

We can observe that, within the vector space V, the reflection with respect to the hyperplane perpendicular to a vector v can be realized by the *minus* conjugate action by v. That is,

Lemma D.1. *Any $v \in V$ such that $\langle v, v \rangle \neq 0$ satisfies, for any $c \in V$,*

$$-vcv^{-1} = c - 2\frac{\langle c, v \rangle}{\langle v, v \rangle} v. \tag{D.3}$$

Proof. $-vcv^{-1} = -vc(\frac{\epsilon}{\langle v,v\rangle} v) = -v\frac{\epsilon}{\langle v,v\rangle}(-vc + 2\epsilon\langle c, v\rangle) = c - 2\frac{\langle c,v\rangle}{\langle v,v\rangle} v.$ □

It is easy to see that this map $V \to V; c \mapsto c - 2\frac{\langle c,v\rangle}{\langle v,v\rangle} v$ for $\langle v, v \rangle \neq 0$ is an element of

$$\mathrm{O}(V) := \{O \in \mathrm{GL}(V) \mid \langle Ou, Ov \rangle = \langle u, v \rangle \text{ for any } u, v \in V\}. \tag{D.4}$$

Based on the above observation, we define

$$P(V) := (\text{the group generated by } \{v \in V \mid \langle v, v \rangle \neq 0\}) \subset \mathrm{Cliff}(V)^\times, \tag{D.5}$$

$$\tilde{P}(V) := \{x \in \mathrm{Cliff}(V)^\times \mid \theta(x) V x^{-1} = V\}, \tag{D.6}$$

where $\mathrm{Cliff}(V)^\times$ denotes the group of all the invertible elements of $\mathrm{Cliff}(V)$. We have $P(V) \subset \tilde{P}(V)$ by Lemma D.1. We also have[50] $\mathbb{K}^\times \subset P(V)$, where $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$, because if we take $v \in V$ such that $\langle v, v\rangle \neq 0$, then any $k \in \mathbb{K}^\times$ can be represented as $k = \frac{\epsilon k}{\langle v,v\rangle} v \cdot v$, where both $\frac{\epsilon k}{\langle v,v\rangle} v$ and v are in $P(V)$. If V is finite-dimensional and $\langle -, -\rangle$ is non-degenerate, we can show that the map

$$\varphi : \tilde{P}(V) \to \mathrm{O}(V); x \mapsto \theta(x) \bullet x^{-1} \tag{D.7}$$

is a group homomorphism satisfying

- $\ker \varphi = \mathbb{K}^\times$,
- $\varphi|_{P(V)} : P(V) \to \mathrm{O}(V)$ is surjective.

Taking all of the above into account, we have an exact sequence

$$1 \to \mathbb{K}^\times \to P(V) = \tilde{P}(V) \xrightarrow{\varphi} O(V) \to 1. \tag{D.8}$$

Pin group $\mathrm{Pin}(V)$

We define the *pin group* $\mathrm{Pin}(V)$ as

$$\mathrm{Pin}(V) := \{v_1 \cdots v_m \mid v_i \in V, \langle v_i, v_i\rangle = \pm 1\} \subset P(V). \tag{D.9}$$

Let us assume that V is finite-dimensional and $\langle -, -\rangle$ is non-degenerate again. If we can normalize any $v \in V$ with $\langle v, v\rangle \neq 0$ so that $\langle tv, tv\rangle = \pm 1$ ($t \in \mathbb{K}^\times$), then $\varphi|_{\mathrm{Pin}(V)} : \mathrm{Pin}(V) \to \mathrm{O}(V)$ is still surjective because $\varphi(v) = \varphi(tv)$. We say that the field $\mathbb{K}$ is *spin* if $\mathbb{K}^\times = (\mathbb{K}^\times)^2 \cup -(\mathbb{K}^\times)^2$ (hence at least one of $t^2 = \langle v, v\rangle^{-1}$ or $t^2 = -\langle v, v\rangle^{-1}$ has a solution $t \in \mathbb{K}^\times$) and the characteristic of $\mathbb{K}$ is not 2. For example, $\mathbb{R}$ and $\mathbb{C}$ are spin. If $\mathbb{K}$ is spin, then we have an exact sequence

$$1 \to F \to \mathrm{Pin}(V) \xrightarrow{\varphi} O(V) \to 1, \tag{D.10}$$

where

$$F = \begin{cases} \mathbb{Z}_2 \cong \{\pm 1\} & (\sqrt{-1} \notin \mathbb{K}), \\ \mathbb{Z}_4 \cong \{\pm 1, \pm\sqrt{-1}\} & (\sqrt{-1} \in \mathbb{K}). \end{cases} \tag{D.11}$$

In addition, under the same assumptions (finite-dimensional V, non-degenerate $\langle -, -\rangle$, spin $\mathbb{K}$), we have[51]

$$\mathrm{Pin}(V) = \{x \in \mathrm{Cliff}(V)^\times \mid \theta(x) V x^{-1} = V, \theta(x)^T x = \pm 1\} \subset \tilde{P}(V), \tag{D.12}$$

[50] [LM89, p. 19] claims that, if we define $\mathbb{K}_0^\times := \mathbb{K}^\times \cap P(V)$, then $\mathbb{K}^\times = \mathbb{K}_0^\times$ or $\mathbb{K}^\times = \mathbb{K}_0^\times \cup (-\mathbb{K}_0^\times)$ holds when $\mathbb{K}$ is spin. However, it seems that $\mathbb{K}^\times = \mathbb{K}_0^\times$ always follows, regardless of whether $\mathbb{K}$ is spin or not, as in the main text.

[51] The details are as follows. It is obvious that $\mathrm{Pin}(V)$ defined in (D.9) is in the set on the right-hand side of (D.12). Conversely, since $P(V) = \tilde{P}(V)$, any element x in the set on the right-hand side of (D.12) can be written in the form of $u_1 \cdots u_l$ ($u_i \in V$) with $q(u_i) \neq 0$. Since $\theta(x)^T x = (-1)^l u_l \cdots u_1 \cdot u_1 \cdots u_l = (-\epsilon)^l \prod_i \langle u_i, u_i\rangle$, the condition $\theta(x)^T x = \pm 1$ translates to $\prod_i \langle u_i, u_i\rangle = \pm 1$. Since $\mathbb{K}$ is spin, we can take $t_i \in \mathbb{K}^\times$ such that $t_i^2 \langle u_i, u_i\rangle = \pm 1$. As a result, $\prod_i t_i^{-2} = \pm 1$, and $\prod_i t_i^{-1} \in F$, where $F \subset \mathrm{Pin}(V)$ is defined in (D.11). Therefore, $x = \prod_i t_i^{-1} \cdot t_1 u_1 \cdots t_l u_l$ is in $\mathrm{Pin}(V)$ defined in (D.9).

where x^T is defined as $x^T = u_m \cdots u_1$ if $x = u_1 \cdots u_m$ ($u_i \in V$), and $x^T = x$ if $x \in \mathbb{R}$. We can show $\theta(x)^T x \in \mathbb{K}^\times$ for any $x \in \tilde{P}(V)$, and it is called the *spinor norm* of x.

Here are some important remarks on the definition of $\mathrm{Pin}(V)$.

- Suppose $\mathbb{K} = \mathbb{R}$. If $\langle -, - \rangle$ is positive- or negative-definite, then we may retain only one corresponding sign of the condition $\langle v_i, v_i \rangle = \pm 1$ of the definition (D.9).

- Suppose $\mathbb{K} = \mathbb{R}$. If $\langle -, - \rangle$ is positive-definite and $\epsilon = 1$, or $\langle -, - \rangle$ is negative-definite and $\epsilon = -1$, then by taking an orthogonal basis $\psi_1, \ldots, \psi_n$ of V with $\langle \psi_i, \psi_i \rangle = \pm 1$, we have $(\psi_i)^2 = +1$ for all i. In this case, $\mathrm{Pin}(V)$ is also written as $\mathrm{Pin}^+(n)$. (D.12) holds even if we replace $\theta(x)^T x = \pm 1$ with $x^T x = +1$.

- Suppose $\mathbb{K} = \mathbb{R}$. If $\langle -, - \rangle$ is positive-definite and $\epsilon = -1$ (e.g. the construction of Duncan's module in these notes), or $\langle -, - \rangle$ is negative-definite and $\epsilon = 1$, then by taking an orthogonal basis as above, we have $(\psi_i)^2 = -1$ for all i. In this case, $\mathrm{Pin}(V)$ is also written as $\mathrm{Pin}^-(n)$. (D.12) holds even if we replace $\theta(x)^T x = \pm 1$ with $\theta(x)^T x = +1$.

- Suppose $\mathbb{K} = \mathbb{C}$. Since we are considering the symmetric bilinear form, not a Hermitian form, there is no concept of the signature of $\langle -, - \rangle$, and we can always take an orthonormal basis $\psi_1, \ldots, \psi_n$ of V with $\langle \psi_i, \psi_i \rangle = +1$. In this case, the pin group defined as in (D.9) is not a double cover of $\mathrm{O}(V)$, as in (D.10, D.11). To avoid it and retain the property that $\mathrm{Pin}(V)$ is a double cover of $\mathrm{O}(V)$, some literature defines (e.g. [Lou01, §17.3])

$$\mathrm{Pin}(V_\mathbb{C}) := \{ v_1 \cdots v_m \mid v_i \in V_\mathbb{C}, \langle v_i, v_i \rangle = +1 \} \tag{D.13}$$
$$= \begin{cases} \{ x \in \mathrm{Cliff}(V_\mathbb{C})^\times \mid \theta(x) V_\mathbb{C} x^{-1} = V_\mathbb{C}, x^T x = +1 \} & (\epsilon = +1) \\ \{ x \in \mathrm{Cliff}(V_\mathbb{C})^\times \mid \theta(x) V_\mathbb{C} x^{-1} = V_\mathbb{C}, \theta(x)^T x = +1 \} & (\epsilon = -1) \end{cases} \tag{D.14}$$
$$\supset \{ \pm \psi_i \}_i, \tag{D.15}$$

or (e.g. [Dun07])

$$\mathrm{Pin}(V_\mathbb{C}) := \{ v_1 \cdots v_m \mid v_i \in V_\mathbb{C}, \langle v_i, v_i \rangle = -1 \} \tag{D.16}$$
$$= \begin{cases} \{ x \in \mathrm{Cliff}(V_\mathbb{C})^\times \mid \theta(x) V_\mathbb{C} x^{-1} = V_\mathbb{C}, \theta(x)^T x = +1 \} & (\epsilon = +1) \\ \{ x \in \mathrm{Cliff}(V_\mathbb{C})^\times \mid \theta(x) V_\mathbb{C} x^{-1} = V_\mathbb{C}, x^T x = +1 \} & (\epsilon = -1) \end{cases} \tag{D.17}$$
$$\supset \{ \pm \sqrt{-1} \psi_i \}_i. \tag{D.18}$$

These $\mathrm{Pin}(V_\mathbb{C})$ are double covers of $\mathrm{O}(V_\mathbb{C})$, that is,

$$1 \to \{\pm 1\} \to \mathrm{Pin}(V_\mathbb{C}) \xrightarrow{\varphi} \mathrm{O}(V_\mathbb{C}) \to 1. \tag{D.19}$$

Spin group $\mathrm{Spin}(V)$

We further define the *spin group* $\mathrm{Spin}(V)$ as

$$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathrm{Cliff}(V)^0 \tag{D.20}$$
$$= \{ v_1 \cdots v_{2l} \mid v_i \in V, \langle v_i, v_i \rangle = \pm 1 \} \tag{D.21}$$
$$= \{ x \in (\mathrm{Cliff}(V)^0)^\times \mid x V x^{-1} = V, x^T x = \pm 1 \}. \tag{D.22}$$

If we define $\mathrm{SO}(V)$ as

$$\mathrm{SO}(V) := \mathrm{O}(V) \cap \mathrm{SL}(V), \tag{D.23}$$

then in parallel with (D.10), we have an exact sequence

$$1 \to F \to \mathrm{Spin}(V) \xrightarrow{\varphi} SO(V) \to 1, \tag{D.24}$$

where F is defined as in (D.11).

Again, here are some important remarks on the definition of $\mathrm{Spin}(n)$.

- Suppose $\mathbb{K} = \mathbb{R}$. If $\langle -, - \rangle$ is positive- or negative-definite, then we may retain only one corresponding sign of the condition $\langle v_i, v_i \rangle = \pm 1$ of (D.21). (D.22) holds even if we replace $x^T x = \pm 1$ with $x^T x = +1$. In this case, $\mathrm{Spin}(V)$ with $\dim V = n$ is also written as $\mathrm{Spin}(n)$, which is a subgroup of $\mathrm{Pin}^{\pm}(n)$.

- Suppose $\mathbb{K} = \mathbb{C}$. In this case, the spin group defined as in (D.20) is not a double cover of $\mathrm{SO}(V)$ again, as in (D.24, D.11). To avoid it and retain the property that $\mathrm{Spin}(V)$ is a double cover of $\mathrm{SO}(V)$, some literature defines $\mathrm{Spin}(V_{\mathbb{C}})$ as the intersection of $\mathrm{Pin}(V_{\mathbb{C}})$ defined in (D.13) or (D.16) and $\mathrm{Cliff}(V)^0$. That is,

$$\begin{aligned} \mathrm{Spin}(V_{\mathbb{C}}) &:= \{v_1 \cdots v_{2l} \mid v_i \in V_{\mathbb{C}}, \langle v_i, v_i \rangle = +1\} && \text{(D.25)} \\ &= \{v_1 \cdots v_{2l} \mid v_i \in V_{\mathbb{C}}, \langle v_i, v_i \rangle = -1\} && \text{(D.26)} \\ &= \{x \in (\mathrm{Cliff}(V_{\mathbb{C}})^0)^{\times} \mid x V_{\mathbb{C}} x^{-1} = V_{\mathbb{C}}, x^T x = +1\}. && \text{(D.27)} \end{aligned}$$

  This $\mathrm{Spin}(V_{\mathbb{C}})$ is a double cover of $\mathrm{SO}(V_{\mathbb{C}})$, that is,

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V_{\mathbb{C}}) \xrightarrow{\varphi} \mathrm{SO}(V_{\mathbb{C}}) \to 1. \tag{D.28}$$

D.2 Explicit Form of Lift of $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$

From now on, we assume V is an n-dimensional $\mathbb{R}$-vector space, and the symmetric bilinear form $\langle -, - \rangle$ on V is positive-definite. We fix the definition of the Clifford algebra as $\mathrm{Cliff}(V) := T(V)/u \otimes u \sim -\langle u, u \rangle$. In this Section D.2, we explicitly describe the lifts of elements of $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$.

To begin with, let us see that $\varphi : \mathrm{Spin}(n) \to \mathrm{SO}(n); x \mapsto x \bullet x^{-1}$ is in fact surjective in this case. Recall that the reflection with respect to the hyperplane perpendicular to a vector $v \in V$ can be realized as in Lemma D.1.

- First, any two-dimensional rotation is a composition of two reflections, which can be shown as follows. Suppose we take a basis of V so that the first two basis vectors span the two-dimensional plane to rotate. Then the matrix representation of the reflection with respect to the hyperplane perpendicular to $\vec{v} = (\cos\theta_v, \sin\theta_v, 0, \dots, 0)$ is

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \mapsto \begin{pmatrix} -2v_1^2 + 1 & -2v_1 v_2 \\ -2v_1 v_2 & -2v_2^2 + 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\cos 2\theta_v & -\sin 2\theta_v \\ -\sin 2\theta_v & \cos 2\theta_v \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{D.29}$$

and the other components are just mapped by the identity matrix. Therefore, the composition of two reflections is

$$\begin{pmatrix} -\cos 2\theta_{v'} & -\sin 2\theta_{v'} \\ -\sin 2\theta_{v'} & \cos 2\theta_{v'} \end{pmatrix} \begin{pmatrix} -\cos 2\theta_{v} & -\sin 2\theta_{v} \\ -\sin 2\theta_{v} & \cos 2\theta_{v} \end{pmatrix} = \begin{pmatrix} \cos 2(\theta_{v'}-\theta_v) & -\sin 2(\theta_{v'}-\theta_v) \\ \sin 2(\theta_{v'}-\theta_v) & \cos 2(\theta_{v'}-\theta_v) \end{pmatrix}. \tag{D.30}$$

- Second, as in the following Lemma D.2, any matrix in $\mathrm{SO}(n)$ can be block-diagonalized into two-dimensional rotation matrices, with a diagonalizing matrix in $\mathrm{SO}(n)$. That is, for any rotation in $\mathrm{SO}(n)$, there is an orthonormal basis such that the whole rotation is a composition of the two-dimensional rotations of the planes spanned by two axes among them.

Lemma D.2. *For any $M \in \mathrm{SO}(n)$, there is $O \in \mathrm{SO}(n)$ which block-diagonalizes M as*

$$O^{-1}MO = \begin{pmatrix} \cos 2\pi\lambda_1 & -\sin 2\pi\lambda_1 & & & \\ \sin 2\pi\lambda_1 & \cos 2\pi\lambda_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi\lambda_{\frac{n}{2}} & -\sin 2\pi\lambda_{\frac{n}{2}} \\ & & & \sin 2\pi\lambda_{\frac{n}{2}} & \cos 2\pi\lambda_{\frac{n}{2}} \end{pmatrix}, \tag{D.31}$$

if n is even, and

$$O^{-1}MO = \begin{pmatrix} \cos 2\pi\lambda_1 & -\sin 2\pi\lambda_1 & & & & \\ \sin 2\pi\lambda_1 & \cos 2\pi\lambda_1 & & & & \\ & & \ddots & & & \\ & & & \cos 2\pi\lambda_{\frac{n-1}{2}} & -\sin 2\pi\lambda_{\frac{n-1}{2}} & \\ & & & \sin 2\pi\lambda_{\frac{n-1}{2}} & \cos 2\pi\lambda_{\frac{n-1}{2}} & \\ & & & & & 1 \end{pmatrix}, \tag{D.32}$$

if n is odd.

Proof. The real Schur decomposition states that there exists $\tilde{O} \in \mathrm{O}(n)$ such that $\tilde{O}^{-1}M\tilde{O}$ is upper quasi-triangular, which means

$$\tilde{O}^{-1}M\tilde{O} = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m} \\ & B_{2,2} & \cdots & B_{2,m} \\ & & \ddots & \vdots \\ & & & B_{m,m} \end{pmatrix}, \tag{D.33}$$

where each $B_{i,j}$ is a 2×2 matrix or a number. Using $(\tilde{O}^{-1}M\tilde{O})^T(\tilde{O}^{-1}M\tilde{O}) = I$, we can further show that $\tilde{O}^{-1}M\tilde{O}$ is quasi-diagonal $\mathrm{diag}(B_{1,1}, \ldots, B_{m,m})$, and $B_{i,i} \in \mathrm{O}(2)$ or $B_{i,i} = \pm 1$. If $B_{i,i} \in \mathrm{O}(2)$ and $\det B_{i,i} = -1$, then such $B_{i,i} \in \mathrm{O}(2)$ can be written as $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, so we diagonalize it to $\mathrm{diag}(1,-1)$ by $\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in \mathrm{SO}(2)$. Then, since $\det(\tilde{O}^{-1}M\tilde{O}) = 1$, there are even number of $B_{i,i} = -1$, and the rest of $B_{i,i}$'s are 1 or elements of $\mathrm{SO}(2)$. Now, by reordering the columns of $\tilde{O}$, we can make $\tilde{O}^{-1}M\tilde{O}$ into the form of (D.31, D.32) within $\tilde{O} \in \mathrm{O}(n)$. If $\det\tilde{O} = -1$, then we just define O as $\tilde{O}$ and obtain the lemma. If $\det\tilde{O} = -1$, then we define O as $\tilde{O}\mathrm{diag}(-1,1,\ldots,1)$, which only changes $B_{1,1}$ from the rotation matrix of angle θ to that of angle $-\theta$, and we obtain the lemma. □

Let $\psi_1,\ldots,\psi_n$ be an orthonormal basis of V. From the above discussions, it is obvious that for any $O \in \mathrm{SO}(n)$ and its lift $\pm\hat{O}$ to $\mathrm{Spin}(n)$,

$$(\pm\hat{O})\left[\begin{pmatrix}\psi_1 \cdots \psi_n\end{pmatrix}\begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix}\right](\pm\hat{O})^{-1} = \begin{pmatrix}\psi_1 \cdots \psi_n\end{pmatrix} O \begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix}, \tag{D.34}$$

where O is represented as a matrix with respect to the basis $\{\psi_i\}_i$.

Let n be even below. Recall that the algebra of $\mathrm{Cliff}(V)$ is

$$\{\psi_i,\psi_j\} = -2\delta_{ij}, \quad (\psi_i)^2 = -1. \tag{D.35}$$

We introduce

$$\Psi_i := \frac{1}{\sqrt{2}}(\psi_{2i-1} + \sqrt{-1}\psi_{2i}), \quad \overline{\Psi}_i := \frac{1}{\sqrt{2}}(\psi_{2i-1} - \sqrt{-1}\psi_{2i}). \tag{D.36}$$

They satisfy

$$\{\Psi_i,\Psi_j\} = \{\overline{\Psi}_i,\overline{\Psi}_j\} = 0, \quad \{\Psi_i,\overline{\Psi}_j\} = -2\delta_{ij}. \tag{D.37}$$

Setting $\theta_{v'} = \pi\lambda_i$ and $\theta_v = 0$ in (D.30), we can see that the lifts of $R_{(0,\ldots,\lambda_i,\ldots,0)} := (\Psi_i \mapsto e^{2\pi\sqrt{-1}\lambda_i}\Psi_i, \Psi_{j\neq i} \mapsto \Psi_j) \in \mathrm{SO}(n)$ are

$$\pm((\cos\pi\lambda_i)\psi_{2i-1} + (\sin\pi\lambda_i)\psi_{2i})\psi_{2i-1} = \mp((\cos\pi\lambda_i)1 + (\sin\pi\lambda_i)\psi_{2i-1}\psi_{2i}) \tag{D.38}$$

$$=: \mp\hat{R}_{(0,\ldots,\lambda_i,\ldots,0)}. \tag{D.39}$$

In particular, if we define

$$r_{2i-1,2i} := \psi_{2i-1}\psi_{2i} \tag{D.40}$$

$$= \frac{\sqrt{-1}}{2}(\Psi_i\overline{\Psi}_i - \overline{\Psi}_i\Psi_i) = \sqrt{-1}(-1 - \overline{\Psi}_i\Psi_i) = \sqrt{-1}(\Psi_i\overline{\Psi}_i + 1), \tag{D.41}$$

then we can see that $(r_{2i-1,2i})^2 = -1$, and therefore

$$\hat{R}_{(0,\ldots,\lambda_i,\ldots,0)} = e^{\pi\lambda_i r_{2i-1,2i}}. \tag{D.42}$$

So, $r_{2i-1,2i}$ corresponds to $2\begin{pmatrix} O & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & O \end{pmatrix}$ in $\mathfrak{so}(n)$.

Another way to check that $e^{\pi\lambda_i r_{2i-1,2i}}$ is a lift of $R_{(0,\dots,\lambda_i,\dots,0)}$ is showing $e^{\pi\lambda_i r_{2i-1,2i}}\Psi_j e^{-\pi\lambda_i r_{2i-1,2i}} = e^{2\pi\sqrt{-1}\lambda_i\delta_{i,j}}\Psi_j$, which follows from $\Psi_i r_{2i-1,2i} = -r_{2i-1,2i}\Psi_i = \sqrt{-1}\Psi_i$.

For more general $M \in \mathrm{SO}(n)$, using Lemma D.2, we block-diagonalize it as

$$O^{-1}MO = R_{\vec{\lambda}} := \begin{pmatrix} \cos 2\pi\lambda_1 & -\sin 2\pi\lambda_1 & & & \\ \sin 2\pi\lambda_1 & \cos 2\pi\lambda_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi\lambda_{\frac{n}{2}} & -\sin 2\pi\lambda_{\frac{n}{2}} \\ & & & \sin 2\pi\lambda_{\frac{n}{2}} & \cos 2\pi\lambda_{\frac{n}{2}} \end{pmatrix}, \tag{D.43}$$

by $O \in \mathrm{SO}(n)$. The lifts of $R_{\vec{\lambda}}$ are

$$\pm\hat{R}_{\vec{\lambda}} := \pm\prod_{i=1}^{\frac{n}{2}}(\cos\pi\lambda_i + \psi_{2i-1}\psi_{2i}\sin\pi\lambda_i) = \pm\prod_{i=1}^{\frac{n}{2}} e^{\pi\lambda_i r_{2i-1,2i}}, \tag{D.44}$$

and therefore the lifts of $M = R^O_{\vec{\lambda}} := OR_{\vec{\lambda}}O^{-1}$ are

$$\pm\hat{R}^O_{\vec{\lambda}} := \pm\hat{O}\hat{R}_{\vec{\lambda}}\hat{O}^{-1} = \pm\prod_{i=1}^{\frac{n}{2}}(\cos\pi\lambda_i + (\hat{O}\psi_{2i-1}\hat{O}^{-1})(\hat{O}\psi_{2i}\hat{O}^{-1})\sin\pi\lambda_i) \tag{D.45}$$

$$= \pm\prod_{i=1}^{\frac{n}{2}}(\cos\pi\lambda_i + (\vec{\psi}O)_{2i-1}(\vec{\psi}O)_{2i}\sin\pi\lambda_i), \tag{D.46}$$

where $\vec{\psi} := (\psi_1, \dots \psi_n)$ is a row vector, and we used (D.34) in the last equation.

E Some Comments on the Stolz–Teichner conjecture

In moonshine phenomena, the modular functions associated with CFTs (the partition functions or elliptic genera) play a central role. In [ST11], a mathematical conjecture which suggests a stronger relation between them was proposed, and moonshine has also begun to be studied in relation to it [GJF18, JF20, Lin22, AKL22]. This conjecture is called the Stolz–Teichner conjecture.

The Stolz–Teichner conjecture says that the ring $\mathrm{TMF}_\bullet$ of the classes of "topological modular forms" is isomorphic to the ring $\mathrm{SQFT}_\bullet$ of the equivalence classes of two-dimensional $\mathcal{N} = (0,1)$ supersymmetric quantum field theories (SQFT). Its precise statement is beyond the scope of these notes, but we add more information as follows.

$\mathrm{TMF}_\bullet$ is a generalized cohomology ring graded by integers $\nu \in \mathbb{Z}$. If we neglect the torsion part of TMF_ν, that is, if we consider $\mathrm{TMF}_\nu \otimes \mathbb{C}$, then it is isomorphic to the space $(\mathrm{MF}_\mathbb{C})^{\text{w-h}}_{\frac{\nu}{2}}$ of the weakly-holomorphic modular forms (Section 3.2) of weight $\frac{\nu}{2}$. Their addition and multiplication correspond to the direct sum and the tensor product of SQFTs. The equivalence relations of SQFTs are defined properly, and for example they include the continuous deformations.

The degree ν of the space SQFT_ν specifies the gravitational anomaly of the theories belonging to it, and if the theory is a two-dimensional $\mathcal{N} = (0,1)$ SCFT of central charge $(c, \tilde{c})$, then $\nu = 2(\tilde{c} - c)$. The map $\mathrm{SQFT}_\bullet \to \mathrm{TMF}_\bullet \to (\mathrm{MF}_\mathbb{Z})^{\text{w-h}}_{\frac{\bullet}{2}}$ associates to a theory $\mathcal{T}$ its elliptic genus[52] normalized by $\eta(\tau)$ as $\eta(\tau)^\nu Z^{\mathcal{T}}_{\text{ell}}(\tau)$. Here, $(\mathrm{MF}_\mathbb{Z})^{\text{w-h}}_{\frac{\bullet}{2}}$ is the ring of weakly holomorphic modular forms with integral q-expansion coefficients. See for example [Tac21, TY23] for more information accessible from the physics side.

If we believe this conjecture, then we can extract nontrivial statements on the space $\mathrm{SQFT}_\bullet$ of SQFTs, by translating the properties of $\mathrm{TMF}_\bullet$. Conversely, verifying such statements serves as a test for the Stolz–Teichner conjecture.

For example, the map $\mathrm{TMF}_\bullet \to (\mathrm{MF}_\mathbb{Z})^{\text{w-h}}_{\frac{\bullet}{2}}$ is not surjective, and an element of its image generated only by $\eta(\tau)$ over $\mathbb{Z}$ is in the form of $\frac{24}{\gcd(24,k)}\eta(\tau)^{24k}$ with k an integer [Hop02, Prop. 4.6]. This suggests that if an $\mathcal{N} = (0,1)$ SCFT with degree $2(\tilde{c} - c) = 24k$ has a constant elliptic

[52] Here, the elliptic genus is in the sense of (3.46). We only deal with CFTs in these notes, but a similar discussion of elliptic genera can be repeated for modular-invariant (up to gravitational anomaly phases) QFTs. The modular transformations of the elliptic genus are [TYY23, §2.1]

$$Z_{\text{ell}}(\tau + 1) = e^{-\pi\sqrt{-1}\frac{\nu}{12}} Z_{\text{ell}}(\tau), \tag{E.1}$$

$$Z_{\text{ell}}(-\frac{1}{\tau}) = e^{-\pi\sqrt{-1}\frac{\nu}{4}} Z_{\text{ell}}(\tau). \tag{E.2}$$

(cf. In the theory of one real free chiral fermion with central charge $c = \frac{1}{2}$, the lowest conformal weight in the R sector is $\frac{1}{16} = \frac{c}{8}$, so $q^{L_0 - \frac{c}{24}} \mapsto e^{\pi\sqrt{-1}\frac{2c}{12}} q^{L_0 - \frac{c}{24}}$ under $\tau \mapsto \tau + 1$.) Here, $Z_{\text{ell}}(\tau) = 0$ unless ν is a multiple of 4, because we obtain $Z_{\text{ell}}(\tau) = e^{-2\pi\sqrt{-1}\frac{\nu}{4}} Z_{\text{ell}}(\tau)$ by applying the modular S transformation twice. So, either $Z_{\text{ell}}(\tau) = 0$, or the phase of the modular S transformation is just a sign. As a result, by combining the elliptic genus with the Dedekind eta function $\eta(\tau)$ (see Appendix B), we can see that $\eta^\nu(\tau) Z(\tau)$ is in fact a modular function of weight $\frac{\nu}{2}$.

genus $Z_{\text{ell}}^{\mathcal{T}}$, then $Z_{\text{ell}}^{\mathcal{T}}$ is divisible by $\frac{24}{\gcd(24,k)}$.

In fact, the Conway moonshine module is such an SCFT with minimal non-zero $k = \pm 1$ and elliptic genus 24. The Conway moonshine module or Duncan's module $V^{f\natural}$ is (the NS sector of) a chiral $\mathcal{N} = 1$ SCFT, whose automorphism group $\mathrm{Aut}_{\mathcal{N}=1}(V^{f\natural})$ preserving its $\mathcal{N} = 1$ superconformal algebra is isomorphic to the sporadic Conway group Co_1. It was first constructed by Duncan in [Dun07], from the theory of 24 real free chiral fermions, through the procedure similar to $\mathbb{Z}_2$-orbifold. So its central charge is $c = \frac{1}{2} \times 24 = 12$. Therefore, by placing it to the right-moving part, and setting the left-moving part trivial,[53] we obtain an $\mathcal{N} = (0,1)$ SCFT belonging to SQFT_{24}. Moreover, the elliptic genus of Duncan's module is a constant (the Witten index; see Section 3.4) thanks to the $\mathcal{N} = 1$ supersymmetry, and its value is 24. As a result, Duncan's module $V^{f\natural}$ is a minimal nontrivial SCFT satisfying the divisibility property suggested by the Stolz–Teichner conjecture.

Another example of implications from the Stolz–Teichner conjecture is our main interest. There exists a class called the periodicity element in $\mathrm{TMF}_{-24^2=-576}$, such that multiplying any class of $\mathrm{TMF}_\bullet$ by it gives rise to a bijection $\mathrm{TMF}_\bullet \to \mathrm{TMF}_{\bullet-576}$. This suggests the existence of an $\mathcal{N} = (0,1)$ SQFT of degree -576, whose elliptic genus is 1.

In [AKL22], a conjectural construction of such an SCFT was proposed based on Duncan's module $V^{f\natural}$. More precisely, they proposed a chiral $\mathcal{N} = 1$ SCFT of central charge $c = \frac{576}{2} = 288$ with a unique vacuum state in the NS sector, with the expectation that its elliptic genus is 1. If we have such a theory, then by regarding it as the left-moving part coupling to the trivial right-moving part, we obtain an SCFT of degree $\nu = 2(0 - 288) = -576$, corresponding to the periodicity element. The explicit construction of such an SCFT is not yet known,[54] so if we can verify that the proposed theory indeed has the elliptic genus 1, then it provides a piece of evidence supporting the

[53]Similarly, if we place Duncan's module to the left-moving part, and set the right-moving part trivial, then we obtain an SCFT belonging to SQFT_{-24}. Of course, its elliptic genus is also 24.

[54]There is also the periodicity element in TMF_{576}. It is known [Dev19] that for any class of TMF_ν with positive degree $\nu > 0$, there exists a corresponding SQFT as an $\mathcal{N} = (0,1)$ sigma model, but the explicit construction of its target manifold is highly nontrivial.

We also note that, if we allow multiple vacua in the NS sector, then a construction of a chiral $\mathcal{N} = 1$ SCFT of central charge $c = 288$ and elliptic genus 1 is known as [AKL22, Eq. (1.5)]

$$24697376(V^{f\natural})^{\otimes 24} \oplus 129179510222461909051548656829595 9(V^{f\natural})^{\otimes 24}/S_{24}. \tag{E.3}$$

Since the elliptic genera of $(V^{f\natural})^{\otimes 24}$ and $(V^{f\natural})^{\otimes 24}/S_{24}$ are constants (the Witten indices) and coprime, these coefficients are found as a solution to the Bézout equation. However, this theory has massively degenerate vacua.

We expect the existence of an SQFT with a unique vacuum corresponding to the periodicity element because of the following facts. First, according to the classification of the chiral fermionic CFTs of central charge $c \leq 24$ [BSLTZ23, Ray23, HM23], every class of TMF_ν with $-48 \leq \nu \leq 0$ is realized by a CFT with a unique vacuum. Second, the above sigma model description guarantees the existence of an SQFT with a unique vacuum for each positive-degree class, so if we had an SQFT corresponding to the periodicity element of degree -576 with a unique vacuum, then by using $\mathrm{TMF}_\bullet \to \mathrm{TMF}_{\bullet-576}$, we can conclude the existence of an SQFT with a unique vacuum for any class of $\mathrm{TMF}_\bullet$.

Stolz–Teichner conjecture. See [JFY24] for another study of the periodicity element in $\mathrm{SQFT}_\bullet$.

The SCFT proposed in [AKL22] is expressed as $(V^{f\natural})^{\otimes 24}/A_{24} \times \mathrm{Co}_1$. Let us see how they arrived at this theory.

Since Duncan's module $V^{f\natural}$ has central charge 12, its 24-fold tensor product $(V^{f\natural})^{\otimes 24}$ indeed has the central charge $c = 12 \times 24 = 288$. The elliptic genus of Duncan's module $V^{f\natural}$ is 24, and thus the elliptic genus of $(V^{f\natural})^{\otimes 24}$ is 24^{24}, which is much larger than 1. Even so, since $V^{f\natural}$ have a quite large symmetry, the Co_1 symmetry, we can try taking the orbifold by this symmetry. Orbifolding is a procedure to make a G-invariant theory $\mathcal{T}/G$ from a theory $\mathcal{T}$ with a finite group symmetry G. It first adds some states to the theory, but then takes the projection onto the G-invariant states, so we can expect the number of states to decrease.

However, we cannot always construct the orbifold theory. There can be an obstruction called the anomaly of the symmetry. One miraculous thing is that the anomaly of the Co_1 symmetry of Duncan's module $V^{f\natural}$ is described as the generator of a group $SH(\mathrm{Co}_1) \cong \mathbb{Z}_{24}$ [JF17, Example 2.4.1], and therefore the anomaly of the diagonal Co_1 symmetry of $(V^{f\natural})^{\otimes 24}$ vanishes. Hence, we can take its orbifold as $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$.

Compared with $24^{24} \sim 1.3 \times 10^{33}$, the order $|\mathrm{Co}_1| \sim 4.2 \times 10^{18}$ of Co_1 is still small, so it is conjectured that we have to take the orbifold by a bigger group. In [AKL22], they calculated that the orbifolds of $(V^{f\natural})^{\otimes 24}$ by the symmetry group S_{24} and its alternating subgroup A_{24}, which act as permutation of the factors of the tensor product, have the elliptic genera $Z_{\mathrm{ell}}^{(V^{f\natural})^{\otimes 24}/S_{24}} = -25499225$ and $Z_{\mathrm{ell}}^{(V^{f\natural})^{\otimes 24}/A_{24}} = 381058359637574 \sim 3.8 \times 10^{14}$. Since these values are smaller than $|\mathrm{Co}_1|$, it is promising to take the orbifold of these theories by the Co_1 symmetry.

In [AKL22], they further claim that the combined symmetry $S_{24} \times \mathrm{Co}_1$ is anomalous[55] (although only S_{24} is non-anomalous). As a result, they finally conjecture that the desired SCFT with elliptic genus 1 is $(V^{f\natural})^{\otimes 24}/A_{24} \times \mathrm{Co}_1$.

Lastly, we remark that the way of constructing an orbifold $\mathcal{T}/G$ from a given theory $\mathcal{T}$ and its symmetry G is not unique, and there exists a degree of freedom called the discrete torsion. In principle, once we construct one of the orbifolds in a consistent way, then all the other orbifolds can be constructed from it (see Section F.1.3).

[55] Let us review their discussion. In the theory of n real free chiral fermions, the group $\mathrm{SO}(n)$ acts on the NS sector as a genuine representation, but it acts on the R sector only as a projective representation; what really acts on the R sector as a genuine representation is its double cover $\mathrm{Spin}(n) \cong \mathbb{Z}_2.\mathrm{SO}(n)$. This appearance of the projective phase of the $\mathrm{SO}(n)$-action on the R sector is regarded as a part of the fermionic anomaly. Similarly, Co_1 genuinely acts on the NS sector $V^{f\natural}$, but it acts on the R sector $V^{f\natural}_{\mathrm{tw}}$ only projectively, and what genuinely acts on $V^{f\natural}_{\mathrm{tw}}$ is $\mathrm{Co}_0 \cong \mathbb{Z}_2.\mathrm{Co}_1$. If this anomaly is vanishing, then Co_1 also acts on $V^{f\natural}_{\mathrm{tw}}$ genuinely, and the action of Co_0 becomes unfaithful because the action of $\mathbb{Z}_2 \subset \mathrm{Co}_0$ becomes trivial. According to [AKL22], the action of Co_0 on the R sector of the orbifold $(V^{f\natural})^{\otimes 24}/S_{24}$ is faithful, so Co_1 can act on this R sector only projectively, which means this $S_{24} \times \mathrm{Co}_1$ symmetry is anomalous. On the other hand, the action of Co_0 on the R sector of $(V^{f\natural})^{\otimes 24}/A_{24}$ is unfaithful.

F Review of Orbifolds

This Appendix F reviews the theoretical foundations of orbifolds. Orbifold is a procedure to make a new theory from a theory with a finite group symmetry, but in general, such a symmetry has an obstruction to the orbifold, called anomaly. In Section F.1, we will review the concept of anomaly, and when and how we can make it vanish. In Section F.2, we will write down the partition function of the orbifold theory, and describe some properties of the twisted partition functions.

F.1 't Hooft Anomaly

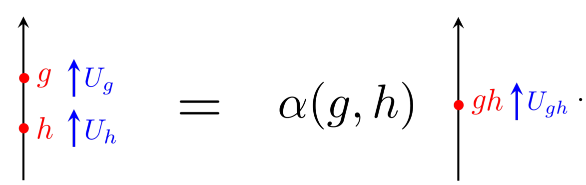
An 't Hooft anomaly [tH80] is a quantum anomaly of a global symmetry G of a theory $\mathcal{T}$. When $\mathcal{T}$ is a bosonic theory of $d+1$ spacetime dimensions with $d = 0, 1, 2$, its 't Hooft anomaly is classified by $H^{d+2}(G; \mathrm{U}(1))$. When it is not trivial, it is an obstruction to orbifolding $\mathcal{T}$ by G. In this Section F.1, we will review it, starting with the case of $0+1$ dimension. At the end, we will describe the anomaly of the Conway symmetry of Duncan's module.

F.1.1 Anomaly in $0+1$ Dimension

Let $\mathcal{T}$ be a $0+1$ dimensional quantum field theory (QFT), namely, a quantum mechanical system, with a finite group symmetry G. The action of $g \in G$ on the Hilbert space $\mathcal{H}$ is implemented by a unitary operator $U_g : \mathcal{H} \to \mathcal{H}$. The group structure requires that the successive implementation of U_g and U_h should be equivalent to U_{gh}, but since the physical states are defined only up to $\mathrm{U}(1)$ phases, an additional phase factor $\alpha(g,h) \in \mathrm{U}(1)$ can appear as

$$U_g U_h = \alpha(g,h) U_{gh}. \tag{F.1}$$

If we draw the one-dimensional timeline, the Hilbert space $\mathcal{H}$ is living on each point, and the action of U_g is depicted as a point operator, as in the following pictorial equation:



(F.2)

The phase α compensates the change of the intermediate picture, while keeping the action of the symmetries from the starting Hilbert space to the ending Hilbert space.

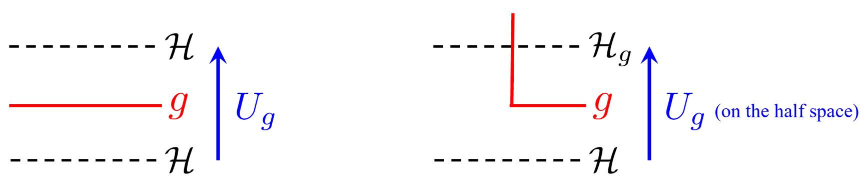
Therefore, G acts on $\mathcal{H}$ as a projective representation, or a representation of an extension of G by $\mathrm{U}(1)$, and its equivalence classes can be classified by the cohomology classes $[\alpha] \in H^2(G; \mathrm{U}(1))$. If we state it more concretely, the associativity $(U_g U_h) U_k = U_g (U_h U_k)$ (or the fact that the operator obtained as the fusion of ordered three operators is independent of the history of changing pictures) imposes the 2-cocycle condition on α, and redefining U_g as $\beta(g) U_g$ ($\beta(g) \in \mathrm{U}(1)$) changes α by the 2-coboundary $d\beta$. In this way, the cohomology class $[\alpha] \in H^2(G; \mathrm{U}(1))$

is associated with the theory $\mathcal{T}$. This is the *'t Hooft anomaly* of the G symmetry of $\mathcal{T}$, and α is referred to as the *anomalous phase*.

When the cohomology class $[\alpha] \in H^2(G; \mathrm{U}(1))$ is trivial, we say that the anomaly is trivial or vanishing, the symmetry is non-anomalous, or the theory is anomaly-free. In such a situation, by the redefinition of U_g, we can make $\alpha(g,h) = 1$ for any $g, h \in G$, so that G acts on $\mathcal{H}$ as a genuine representation. Otherwise, the 'projection operator' $P := \frac{1}{|G|}\sum_{g\in G} U_g$ onto G-invariant states is not truly the projection, for example because $U_g P \neq P$. Analogously, a nontrivial anomaly in $1+1$ dimensions is an obstruction to orbifolding or gauging the theory, which is a procedure to make a G-invariant theory.

F.1.2 Anomaly in $1+1$ Dimensions

Next, let $\mathcal{T}$ be a 1+1 dimensional QFT with a finite group symmetry[56] G. On the two-dimensional spacetime, the group action $U_g : \mathcal{H} \to \mathcal{H}$ on the time-sliced Hilbert space $\mathcal{H}$ is depicted as a horizontal line. Moreover, if the symmetry action is local, then we may perform it only on a segment or a half-line of the entire one-dimensional space. As a result, the defect, or the twisted boundary condition, appears at the boundary, and it is depicted as a vertical line (see Figure F.1). We can also define more curved lines by moving the boundary (see for example [Sei23]). We will write the g-twisted Hilbert space, realized by the action of g on the right-half space, as $\mathcal{H}_g$. If there are multiple twisted boundary conditions $g_1, \ldots, g_n$ inserted, then the twisted Hilbert space is denoted by $\mathcal{H}_{g_1,\ldots,g_n}$.



(a) (b)

Figure F.1: (a) The action $U_g : \mathcal{H} \to \mathcal{H}$ of $g \in G$ depicted as a horizontal line. (b) The twisted Hilbert space implemented by the half-space action.

The fusion of such lines is implemented by the fusion operator

$$u_{g,h} : \mathcal{H}_{g,h} \to \mathcal{H}_{gh}. \tag{F.3}$$

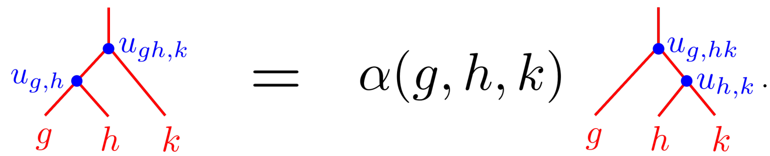
[56] As seen in Section F.1.1, a symmetry G of a $0+1$ dimensional theory acts on the Hilbert space as a projective representation. In $1+1$ dimension, on the other hand, when we say that the theory has a symmetry G, we assume that G acts on the untwisted Hilbert space as a genuine representation. Even so, it can acts on the twisted Hilbert spaces as projective representations.

For example, the automorphism group $\mathrm{Aut}(V)$ of a VOA V acts on the untwisted Hilbert space V as a genuine representation, by its mathematical definition. For a monster VOA $V^\natural$, the anomaly of $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$ is known to have order 24 in $H^3(\mathbb{M}; \mathrm{U}(1))$ [JF17, Thm. 1]. So it is conjectured that $H^3(\mathbb{M}; \mathrm{U}(1)) \cong \mathbb{Z}_{24}$ in [JF17].

The order of the fusion of three lines $\mathcal{H}_{g,h,k} \to \mathcal{H}_{ghk}$ should not affect the physics, which implies the existence of a phase factor $\alpha(g,h,k) \in \mathrm{U}(1)$ such that

$$u_{gh,k} u_{g,h} = \alpha(g,h,k) u_{g,hk} (U_g u_{h,k} U_g^{-1}), \tag{F.4}$$

where $U_g \bullet U_g^{-1}$ appears because we implemented the g-twist on the time-sliced Hilbert space by letting U_g act on the right-half space. As a pictorial equation,



(F.5)

Again, the phase α compensates the change of the intermediate picture, while keeping the action of the symmetries from the starting Hilbert space to the ending Hilbert space.

We omit the details, but the pentagon identity (the fact that the final picture obtained by changing the order of fusions from the given picture is independent of the history of changing pictures; see Figure F.2) imposes the 3-cocycle condition on α, and redefining $u_{g,h}$ as $\beta(g,h)u_{g,h}$ ($\beta(g,h) \in \mathrm{U}(1)$) changes α by the 3-coboundary $d\beta$.

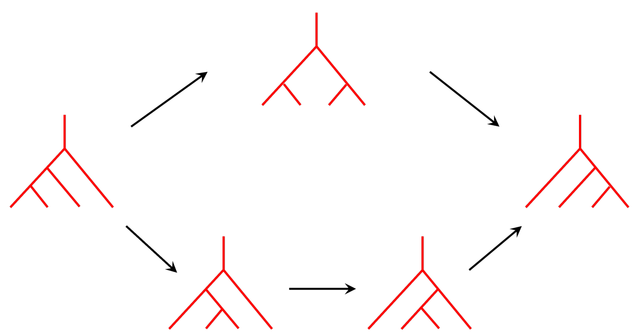


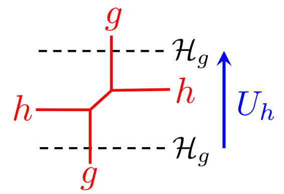
Figure F.2: The change of pictures to obtain the pentagon identity.

In this way, the cohomology class $[\alpha] \in H^3(G; \mathrm{U}(1))$ is associated with the theory $\mathcal{T}$. This fact was known in the context of algebraic QFT [Mug04], and also established in the context of condensed matter physics in [EN14]; see also [Sei23]. This is the 't Hooft anomaly of the G symmetry of $\mathcal{T}$.

F.1.3 Canceling Anomaly and Discrete Torsion

As we will see in Section F.2, the orbifold of a $(1+1)$-dimensional CFT $\mathcal{T}$ by G is constructed by adding the twisted sectors and projecting them onto the G-invariant states. Assume that we define the action of U_h on the twisted sector $\mathcal{H}_g$, where g and h commute, as

$$U_h \;:=$$

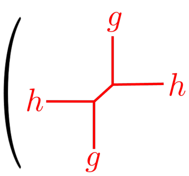


(F.6)

(If g and h do not commute, U_h maps $\mathcal{H}_g$ to $\mathcal{H}_{hgh^{-1}}$.) If the anomalous phases α in (F.5) are nontrivial, then the composition $U_h U_k$ of actions on $\mathcal{H}_g$ differs from U_{hk} by a $\mathrm{U}(1)$ phase, so we cannot define the appropriate projection operator as $\frac{1}{|G|}\sum_{g\in G} U_g$, for a similar reason to the case of $0+1$ dimension in Section F.1.1. This is an obstruction to orbifolding.

Moreover, a nontrivial anomaly is also an obstruction to the modular invariance of the orbifold. Assume that we define the twisted partition function $Z_g^h(\tau)$ on a torus for commuting g and h as

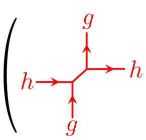
$$Z_g^h(\tau) \;:=\; Z\left(\vphantom{\int}\right.$$



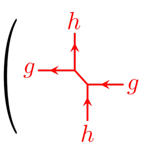
$$\left.\vphantom{\int}\right). \qquad \text{(F.7)}$$

Then, the anomalous phases appear in the modular transformations, say the modular S transformation, as

$$Z_g^h(\tau) = Z\left(\vphantom{\int}\right.$$



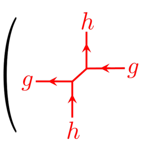
$$\left.\vphantom{\int}\right) \overset{S}{\longmapsto} Z\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right)$$

$$\Vert$$

$$(\text{phase from } \alpha)\; Z\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = (\text{phase from } \alpha)\; Z_h^{g^{-1}}(\tau)\,. \qquad \text{(F.8)}$$

Such phases break the modular invariance property of the orbifold theory, unless the anomalous phases α are all trivial.

If the cohomology class $[\alpha] \in H^3(G;\mathrm{U}(1))$ is trivial, then by taking a 2-cochain β_0 such that $\alpha(g,h,k) = d\beta_0(g,h,k) = \beta_0(h,k)\beta_0(gh,k)^{-1}\beta_0(g,hk)\beta_0(g,h)^{-1}$ and rewriting (F.4) using a new fusion operator $(u_0)_{g,h} := \beta_0(g,h)u_{g,h}$, we can remove the anomalous phase α from (F.4), so that the obstacle to a modular-invariant orbifold theory does not exist. In particular, if we write the twisted partition function defined as in (F.7) using fusion operators $u_{g,h}$ as $Z^{(u)}{}_g^h(\tau)$, then the new twisted partition function is

$$Z^{(u_0)}{}_g^h(\tau) = \frac{\beta_0(h,g)}{\beta_0(g,h)} Z^{(u)}{}_g^h(\tau), \qquad \text{(F.9)}$$

and this $Z^{(u_0)}{}^h_g(\tau)$ is the twisted partition function we can actually use for the computation of the twisted partition function of the orbifold theory.

However, there is a degree of freedom in the choice of the 2-cochain β_0 which is a solution to the equation $\alpha = d\beta_0$. In fact, even if we multiply one solution β_0 by a 2-cocycle β, the resulting $\beta_0 \cdot \beta$ is also a solution.[57] If we define another new fusion operator $(u_0')_{g,h} := \beta_0(g,h)\beta(g,h)u_{g,h}$, then the resulting twisted partition function $Z^{(u_0')}{}^h_g(\tau)$ is different from the originally modified one $Z^{(u_0)}{}^h_g(\tau)$ by $\frac{\beta(h,g)}{\beta(g,h)}$ in general. When the 2-cocycle β is just a 2-coboundary $\beta(g,h) = d\gamma(g,h) = \gamma(h)\gamma(gh)^{-1}\gamma(g)$, then this difference $\frac{\beta(h,g)}{\beta(g,h)}$ is just 1 because g and h commute, and hence $Z^{(u_0')}{}^h_g(\tau)$ is the same as $Z^{(u_0)}{}^h_g(\tau)$.

As a result, we can say that if there is one way of canceling the anomaly (using β_0), then there are $\#H^2(G;\mathrm{U}(1))$ different ways of canceling the anomaly (using $\beta_0\beta$ for $[\beta] \in H^2(G;\mathrm{U}(1))$) leading to different twisted partition functions, and hence different orbifold theories. This degree of freedom is called the *discrete torsion*.

F.1.4 Anomaly in Fermionic Systems

Finally, we briefly mention the anomaly in fermionic systems. Let us consider a fermionic theory with a finite group symmetry G. We demand that the G-action should preserve the parity, that is, it should commute with the symmetry $\mathbb{Z}_2 = \langle(-1)^F\rangle$ defining the fermionic parity.[58] (As a remark, in mathematics, the definition of an automorphism of a VOSA contains the condition that it preserves the parity.) We also assume that G acts on the NS sector as a genuine representation, but G may act on the R sector as a projective representation, and it is regarded as a part of the fermionic anomaly. A prototypical example[59] is that, $\mathrm{SO}(n)$ acts on the NS sector $A(\mathfrak{a})$ of n real free fermions as a genuine representation, but it acts on the R sector $A(\mathfrak{a})_{\mathrm{tw}}$ only as a projective representation. This is because what really acts on $A(\mathfrak{a})_{\mathrm{tw}}$ as a genuine representation is its double cover $\mathrm{Spin}(n) = \mathbb{Z}_2.\mathrm{SO}(n)$; the projective $\mathrm{SO}(n)$-action on the R sector is the composition of a section $s : \mathrm{SO}(n) \to \mathrm{Spin}(n)$ and the genuine $\mathrm{Spin}(n)$-action.[60]

Whereas the bosonic anomalies in $1+1$ dimensions are classified by the ordinary cohomology

[57] Since U(1) is a multiplicative group, the linearity of the differential d is $d(\beta_0 \cdot \beta) = d\beta_0 \cdot d\beta$, and the cocycle condition is $d\beta = 1$.

[58] A fermionic theory $\mathcal{T}[\sigma]$ on a space-time manifold M depends on the spin structure σ on M. The set of spin structures is an affine space modeled on $H^1(M;\mathbb{Z}_2)$, so by introducing a $\mathbb{Z}_2$ gauge field $A \in H^1(M;\mathbb{Z}_2)$ as $\mathcal{T}[\sigma, A] := \mathcal{T}[\sigma + A]$ up to anomalous phases [BSZ24], we obtain the canonical $\mathbb{Z}_2$ symmetry $\langle(-1)^F\rangle$ of the fermionic theory. Then the assumption that G commutes with $(-1)^F$ also means that the G-action does not change the NS sector to the R sector, and the R sector to the NS sector. See also footnote 34.

Note that, in [DGG21], the action of a symmetry G is assumed not to contain the change of spin structure σ (Be careful that this change seems denoted by the action by $(-1)^F$ there, and $G_f = \langle(-1)^F\rangle.G$), but by absorbing the change of σ into the change of the gauge field A, we can also apply the discussion there to G containing $(-1)^F$.

[59] Another example is that, $V^{f\natural}$ admits the action by $\mathrm{SemiSpin}(24)$ (or $\mathrm{Co}_1 \in \mathrm{SemiSpin}(24)$ if we take the $\mathcal{N} = 1$ structure into account), but it acts on $V^{f\natural}_{\mathrm{tw}}$ only as a projective representation, because what really acts on $V^{f\natural}_{\mathrm{tw}}$ as a genuine representation is its double cover $\mathrm{Spin}(24)$ (or $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$).

[60] Since $\mathrm{SO}(n)$ is a continuous group, we cannot apply the discussion of Section F.1.2. In particular, we cannot naively say that "the anomaly of $\mathrm{SO}(n)$ is classified by $H^3(\mathrm{SO}(n);\mathrm{U}(1))$."

$H^3(G;\mathrm{U}(1))$, the fermionic anomalies in $1+1$ dimensions are classified by the supercohomology $SH^3(G)$ [KT17, WG17]. As a set, $SH^3(G)$ is the same as $H^3(G;\mathrm{U}(1))\oplus H^2(G;\mathbb{Z}_2)\oplus H^1(G;\mathbb{Z}_2)$. The first layer $H^3(G;\mathrm{U}(1))$ is coming from the anomalies similar to the bosonic cases. The second layer $H^2(G;\mathbb{Z}_2)$ is coming from the fact that the fusion operators $u_{g,h}$ can have fermionic parities in the fermionic cases (see e.g. [EN14, Tac18, OSTZ25]), and sometimes called the Gu-Wen layer [GW12]. The third layer $H^1(G;\mathbb{Z}_2)$ describes anomalies which occur when the number of Majorana fermions is odd, for example at the edge of the Kitaev chain [Kit00].

As an example, the anomaly of a $\mathbb{Z}_2$ symmetry of one real fermion in $1+1$ dimensions corresponds to the generator of $SH(\mathbb{Z}_2) = H^3(\mathbb{Z}_2;\mathrm{U}(1)).H^2(\mathbb{Z}_2;\mathbb{Z}_2).H^1(\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2.\mathbb{Z}_2.\mathbb{Z}_2 \cong \mathbb{Z}_8$. See [DGG21] for more details.

In higher $d+1$ dimensions, the fermionic anomalies are classified by a certain generalized cohomology theory (the Anderson dual of the spin bordism group $\Omega^{\mathrm{spin}}_{d+2}(BG)$) [KTTW14, FH16, GJF17, Yon18], and the supercohomology $SH^{d+2}(G)$ only captures the first three layers of it.

In the case of the Co_1 symmetry of Duncan's module $V^{f\natural}$, it is known [JF17, Example 2.4.1] that its anomaly is the generator of $SH^3(\mathrm{Co}_1) = H^3(\mathrm{Co}_1;\mathrm{U}(1)).H^2(\mathrm{Co}_1;\mathbb{Z}_2) \cong \mathbb{Z}_{12}.\mathbb{Z}_2 \cong \mathbb{Z}_{24}$. Therefore, if we take the 24-fold tensor product of $V^{f\natural}\oplus V^{f\natural}_{\mathrm{tw}}$, its diagonal Co_1-action is non-anomalous, and we can consider the Conway orbifold theory $(V^{f\natural}\oplus V^{f\natural}_{\mathrm{tw}})^{\otimes 24}/\mathrm{Co}_1$. Recall that the $\mathbb{Z}_2$ symmetry defining the fermion parity of $V^{f\natural}\oplus V^{f\natural}_{\mathrm{tw}}$ is $\langle -1_{\mathrm{SemiSpin}(24)}\rangle$. Since $-1_{\mathrm{SemiSpin}(24)}$ is not contained in $\mathrm{Co}_1 \subset \mathrm{SemiSpin}(24)$, this orbifold does not mix the NS sector and the R sector (see footnote 58).

Remark F.1. The anomaly of the Co_0 symmetry of $V^{s\natural}$ is also known [JF17, Example 2.4.1] to be the generator of $SH^3(\mathrm{Co}_0) = H^3(\mathrm{Co}_0;\mathrm{U}(1)) \cong \mathbb{Z}_{24}$. (In fact, we have seen that $H^2(\mathrm{Co}_0;\mathbb{Z}_2)$ is trivial in footnote 37.) So we can repeat the similar arguments for $V^{s\natural}$. *(Remark ends.)*

Remark F.2. Since $SH^3(\mathrm{Co}_0) \cong \mathbb{Z}_{24}$, we may consider the orbifold of $(V^{f\natural}\oplus V^{f\natural}_{\mathrm{tw}})^{\otimes 24}$ by the preimage $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$ of $\mathrm{Co}_1 \subset \mathrm{SemiSpin}(24)$. The action of Co_0 on $V^{f\natural}\oplus V^{f\natural}_{\mathrm{tw}}$ is the same as that of Co_1 (see footnote 59), but the number of twisted sectors are different (Co_1 has 101 conjugacy classes, whereas Co_0 has 167 conjugacy classes), so $(V^{f\natural}\oplus V^{f\natural}_{\mathrm{tw}})^{\otimes 24}/\mathrm{Co}_1$ and $(V^{f\natural}\oplus V^{f\natural}_{\mathrm{tw}})^{\otimes 24}/\mathrm{Co}_0$ are different theories. *(Remark ends.)*

F.2 Orbifold Partition Function

If a CFT $\mathcal{T}$ has a non-anomalous finite group symmetry G, we can construct a new CFT $\mathcal{T}/G$ called the orbifold of $\mathcal{T}$ by G, consisting of G-invariant states. In this Section F.2, we will review it, focusing on its partition function. After reviewing the construction of orbifold in Section F.2.1, we will reduce the expression of its partition function, using the properties of the action of G on twisted sectors in Section F.2.2. We will investigate the $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{GL}(2,\mathbb{Z})$ transformations of the twisted partition functions in Section F.2.3.

F.2.1 Basic Construction of Orbifold

Let $\mathcal{T}$ be a two-dimensional CFT on a torus with a non-anomalous finite group symmetry G. The *orbifold* $\mathcal{T}/G$ of $\mathcal{T}$ by G is constructed by the following two steps (see e.g. [Pol07, §8.5]).

1. Add all the twisted sectors $\mathcal{H}_g$ ($g \in G$) and define

$$\mathcal{H}_{\text{tot}} := \bigoplus_{g \in G} \mathcal{H}_g, \tag{F.10}$$

   where $\mathcal{H}_1$ is the Hilbert space of the original theory $\mathcal{T}$.

2. Define the Hilbert space $\mathcal{H}^{(\mathcal{T}/G)}$ of the orbifold $\mathcal{T}/G$ as the G-invariant states of $\mathcal{H}_{\text{tot}}$. That is,

$$\mathcal{H}^{(\mathcal{T}/G)} := \left(\frac{1}{|G|} \sum_{g \in G} U_g \right) \mathcal{H}_{\text{tot}}. \tag{F.11}$$

We define the *twisted partition function* of $\mathcal{T}$, with spatial twist g and temporal twist h as

$$Z^{(\mathcal{T})}{}^h_g(\tau) := \mathrm{Tr}_{\mathcal{H}_g}[U_h q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}] \quad (q := e^{2\pi\sqrt{-1}\tau}), \tag{F.12}$$

where $U_h : \mathcal{H}_{\text{tot}} \to \mathcal{H}_{\text{tot}}$ is the unitary action of $h \in G$, and $\tilde{}$ denotes the right-moving part, although we will discuss chiral theories with only left-moving parts after this Section F.2. Recall that, in general, we need the phase modification (F.9) to make the anomalous phases trivial. Let $Z_0^{(\mathcal{T})}{}^h_g(\tau)$ denote the twisted partition function after the phase modification. The partition function of the orbifold $\mathcal{T}/G$ is then

$$Z^{(\mathcal{T}/G)}(\tau) = \frac{1}{|G|} \sum_{g,h \in G} Z_0^{(\mathcal{T})}{}^h_g(\tau). \tag{F.13}$$

In the rest of this Section F.2, we will investigate the properties of the above concepts, and rewrite the orbifold partition function (F.13) so that the number of summands will be reduced.

F.2.2 Twisted Partition Functions As Class Functions

We have added the twisted sectors. In the field description, a field $\phi(\sigma^1, \sigma^2)$ belonging to the g-twisted sector $\mathcal{H}_g$ satisfies

$$\phi(\sigma^1 + 2\pi, \sigma^2) = g \cdot \phi(\sigma^1, \sigma^2). \tag{F.14}$$

Then we can see that the action $U_h : \mathcal{H}_{\text{tot}} \to \mathcal{H}_{\text{tot}}$ of $h \in G$ restricted to $\mathcal{H}_g$ is

$$U_h : \mathcal{H}_g \to \mathcal{H}_{hgh^{-1}}, \tag{F.15}$$

because if $\phi(\sigma^1, \sigma^2)$ belongs to the g-twisted sector, then $\phi'(\sigma^1, \sigma^2) := h \cdot \phi(\sigma^1, \sigma^2)$ belongs to the hgh^{-1}-twisted sector as

$$\phi'(\sigma^1 + 2\pi, \sigma^2) = hg\phi(\sigma^1, \sigma^2) = hgh^{-1}\phi'(\sigma^1, \sigma^2). \tag{F.16}$$

This fact leads to the following two consequences on the twisted partition functions.

(1) Since $U_h : \mathcal{H}_g \to \mathcal{H}_{hgh^{-1}}$,

$$Z_0^{(\mathcal{T})}{}^h_g(\tau) = \mathrm{Tr}_{\mathcal{H}_g}[U_h q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}] = 0 \quad \text{if } hg \neq gh. \tag{F.17}$$

In other words, $\mathcal{H}_g$ is a representation of the centralizer $C_g := \{h \in G \mid hg = gh\}$ of g. (If G is anomalous, then $\mathcal{H}_g$ is just a projective representation of C_g.)

(2) The representations of the chiral algebra $\mathcal{A}$ on $\mathcal{H}_g$ and $\mathcal{H}_{hgh^{-1}}$ are equivalent under U_h. That is, if we write the representation of $\mathcal{A}$ on $\mathcal{H}_g$ as $\rho_g : \mathcal{A} \to \mathrm{End}(\mathcal{H}_g)$, then

$$U_h \rho_g(\bullet) U_h^{-1} = \rho_{hgh^{-1}}(\bullet). \tag{F.18}$$

Therefore,

$$\begin{aligned} Z_0^{(\mathcal{T})}{}^h_{kgk^{-1}}(\tau) &= \mathrm{Tr}_{\mathcal{H}_{kgk^{-1}}}[U_h q^{\rho_{kgk^{-1}}(L_0) - \frac{c}{24}} \bar{q}^{\tilde{\rho}_{kgk^{-1}}(\tilde{L}_0) - \frac{\tilde{c}}{24}}] && \text{(F.19)}\\ &= \mathrm{Tr}_{\mathcal{H}_{kgk^{-1}}}[U_h U_k q^{\rho_g(L_0) - \frac{c}{24}} \bar{q}^{\tilde{\rho}_g(\tilde{L}_0) - \frac{\tilde{c}}{24}} U_k^{-1}] && \text{(F.20)}\\ &= \mathrm{Tr}_{\mathcal{H}_g}[U_{k^{-1}} U_h U_k q^{\rho_g(L_0) - \frac{c}{24}} \bar{q}^{\tilde{\rho}_g(\tilde{L}_0) - \frac{\tilde{c}}{24}}] && \text{(F.21)}\\ &= Z_0^{(\mathcal{T})}{}^{k^{-1}hk}_g(\tau), && \text{(F.22)} \end{aligned}$$

where we used the cyclicity of the trace in the third equation. In other words, the twisted partition functions are class functions in the sense that

$$Z_0^{(\mathcal{T})}{}^h_g(\tau) = Z_0^{(\mathcal{T})}{}^{khk^{-1}}_{kgk^{-1}}(\tau). \tag{F.23}$$

Combining (F.17) and (F.23), the twisted partition functions $Z_0^{(\mathcal{T})}{}^h_g(\tau)$ can be regarded as a function over the conjugacy classes of the commuting pairs

$$G^2_{\mathrm{com}}/\!\sim\; := \{(g,h) \in G \times G \mid gh = hg\}/(g,h) \sim (kgk^{-1}, khk^{-1}). \tag{F.24}$$

Let $G/\!\sim\; := G/(g \sim kgk^{-1})$ denote the set of conjugacy classes, and $[g] \in G/\!\sim$ be the conjugacy class of $g \in G$. We introduce the notation $\sum_{[g]\in G/\sim}$ to represent the sum $\sum_{g\in\{g_1,\ldots,g_r\}}$ over a complete set $\{g_1, \ldots, g_r\}$ of representatives of the quotient $G/\!\sim\; = \{[g_1], \ldots, [g_r]\}$. When this notation is used, the sum is supposed to be independent of the choice of the representatives. Now, using

$$\sum_{h\in G} Z_0^{(\mathcal{T})}{}^h_{kgk^{-1}}(\tau) = \sum_{h\in G} Z_0^{(\mathcal{T})}{}^{k^{-1}hk}_g(\tau) = \sum_{h\in G} Z_0^{(\mathcal{T})}{}^h_g(\tau), \tag{F.25}$$

the orbifold partition function is

$$Z^{(\mathcal{T}/G)}(\tau) = \frac{1}{|G|}\sum_{g\in G}\sum_{h\in G} Z_0^{(\mathcal{T})}{}^h_g(\tau) \tag{F.26}$$

$$= \frac{1}{|G|}\sum_{[g]\in G/\sim} |[g]| \sum_{h\in G} Z_0^{(\mathcal{T})}{}^h_g(\tau) \tag{F.27}$$

$$= \sum_{[g]\in G/\sim} \frac{1}{|C_g|} \sum_{h\in G} Z_0^{(\mathcal{T})}{}^h_g(\tau) \tag{F.28}$$

$$= \sum_{[g]\in G/\sim} \frac{1}{|C_g|} \sum_{h\in C_g} Z_0^{(\mathcal{T})}{}^h_g(\tau), \tag{F.29}$$

where in the third equation, we used $|G| = |[g]||C_g|$.

We can further decompose the centralizer C_g into conjugacy classes $[h]_{\subset C_g} \in C_g/\sim$. Then, since $Z_0^{(\mathcal{T})}{}^{khk^{-1}}_g(\tau) = Z_0^{(\mathcal{T})}{}^h_{k^{-1}gk}(\tau) = Z_0^{(\mathcal{T})}{}^h_g(\tau)$ for $k \in C_g$,

$$Z^{(\mathcal{T}/G)}(\tau) = \sum_{[g]\in G/\sim} \sum_{[h]_{\subset C_g}\in C_g/\sim} \frac{|[h]_{\subset C_g}|}{|C_g|} Z_0^{(\mathcal{T})}{}^h_g(\tau). \tag{F.30}$$

Finally, recall that $Z_{g,h}(\tau)$ can be regarded as a function over $G^2_{\text{com}}/\sim$ defined in (F.24). The summands of (F.30) are labeled by[61] $[h]_{\subset C_g} \in \bigsqcup_{[g]\in G/\sim} C_g/\sim$. It is straightforward to show that the map $[h]_{\subset C_g} \mapsto [(g,h)]$ defines a well-defined[62] bijection $\bigsqcup_{[g]\in G/\sim} C_g/\sim \to G^2_{\text{com}}/\sim$. Therefore, we have

$$Z^{(\mathcal{T}/G)}(\tau) = \sum_{[(g,h)]\in G^2_{\text{com}}/\sim} \frac{|[h]_{\subset C_g}|}{|C_g|} Z_0^{(\mathcal{T})}{}^h_g(\tau). \tag{F.31}$$

F.2.3 Modular Transformations of Twisted Partition Functions

Under the modular transformations, the twisted partition functions transform as (see e.g. [Pol07, §8.5])

$$Z_0^{(\mathcal{T})}{}^h_g(\tau+1) = Z_0^{(\mathcal{T})}{}^{gh}_g(\tau), \tag{F.32}$$

$$Z_0^{(\mathcal{T})}{}^h_g(-\frac{1}{\tau}) = Z_0^{(\mathcal{T})}{}^{g^{-1}}_h(\tau), \tag{F.33}$$

if the phases caused by the gravitational anomaly in the modular transformations (E.1, E.2) are trivial, that is, $2(\tilde{c}-c) \equiv 0 \bmod 24$; the gravitational anomaly phases cannot be canceled by the phase modification (F.9) to cancel the 't Hooft anomaly. It then follows that the orbifold partition function $Z^{(\mathcal{T}/G)}(\tau) = \sum_{g,h\in G} Z_0^{(\mathcal{T})}{}^h_g(\tau)$ is modular invariant.

[61] Although the set $\bigsqcup_{[g]\in G/\sim} C_g/\sim := (C_{g_1}/\sim) \sqcup \cdots \sqcup (C_{g_r}/\sim)$ depends on the choice of the representatives $\{g_1,\ldots,g_r\}$ of $G/\sim$, the sum (F.30) is independent of such a choice.

[62] It is well-defined in the sense that if $[h]_{\subset C_g} = [h']_{\subset C_g}$ then $[(g,h)] = [(g,h')]$.

Modular transformations of fermionic twisted partition functions

Assume that the theory $\mathcal{T}$ is fermionic and let $\langle(-1)^F\rangle$ denote the $\mathbb{Z}_2$ symmetry defining the fermion parity. We demand that the symmetry G should commute with $(-1)^F$. We then define the twisted partition functions for $g, h \in G$ as

$$Z^{(\mathcal{T})}{}^{\mathrm{NS}h}_{\mathrm{NS}g}(\tau) := \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}g}}[U_h q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}], \tag{F.34}$$

$$Z^{(\mathcal{T})}{}^{\mathrm{R}h}_{\mathrm{NS}g}(\tau) := \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}g}}[(-1)^F U_h q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}], \tag{F.35}$$

$$Z^{(\mathcal{T})}{}^{\mathrm{NS}h}_{\mathrm{R}g}(\tau) := \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}g}}[U_h q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}], \tag{F.36}$$

$$Z^{(\mathcal{T})}{}^{\mathrm{R}h}_{\mathrm{R}g}(\tau) := \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}g}}[(-1)^F U_h q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}], \tag{F.37}$$

and again, let $Z_0^{(\mathcal{T})}{}^{\mathrm{Y}h}_{\mathrm{X}g}(\tau)$ $(\mathrm{X},\mathrm{Y} \in \{\mathrm{NS},\mathrm{R}\})$ denote the ones after the phase modifications (F.9), when G is non-anomalous.

The modular transformations of these twisted partition functions can be derived by the path integral formalism on a torus as follows. We have seen in footnote 30 that, in the cylinder coordinates, the NS sector is anti-periodic and the R sector is periodic in the spatial direction. In addition, unlike the bosonic path integral, the fermionic path integral which is periodic in temporal direction corresponds to the trace with $(-1)^F$ inserted, and the anti-periodic one corresponds to the usual trace [Pol07, Appendix A.2]. As a result, we may regard NS and R in (F.34)–(F.37) as the nontrivial element and the identity element of $\mathbb{Z}_2 = \langle(-1)^F\rangle$, respectively. This discussion leads to the modular transformations of the fermionic twisted partition functions as

$$Z_0^{(\mathcal{T})}{}^{\mathrm{NS}h}_{\mathrm{NS}g}(\tau+1) = Z_0^{(\mathcal{T})}{}^{\mathrm{R}gh}_{\mathrm{NS}g}(\tau), \qquad Z_0^{(\mathcal{T})}{}^{\mathrm{NS}h}_{\mathrm{NS}g}(-\frac{1}{\tau}) = Z_0^{(\mathcal{T})}{}^{\mathrm{NS}g^{-1}}_{\mathrm{NS}h}(\tau), \tag{F.38}$$

$$Z_0^{(\mathcal{T})}{}^{\mathrm{R}h}_{\mathrm{NS}g}(\tau+1) = Z_0^{(\mathcal{T})}{}^{\mathrm{NS}gh}_{\mathrm{NS}g}(\tau), \qquad Z_0^{(\mathcal{T})}{}^{\mathrm{R}h}_{\mathrm{NS}g}(-\frac{1}{\tau}) = Z_0^{(\mathcal{T})}{}^{\mathrm{NS}g^{-1}}_{\mathrm{R}h}(\tau), \tag{F.39}$$

$$Z_0^{(\mathcal{T})}{}^{\mathrm{NS}h}_{\mathrm{R}g}(\tau+1) = Z_0^{(\mathcal{T})}{}^{\mathrm{NS}gh}_{\mathrm{R}g}(\tau), \qquad Z_0^{(\mathcal{T})}{}^{\mathrm{NS}h}_{\mathrm{R}g}(-\frac{1}{\tau}) = Z_0^{(\mathcal{T})}{}^{\mathrm{R}g^{-1}}_{\mathrm{NS}h}(\tau), \tag{F.40}$$

$$Z_0^{(\mathcal{T})}{}^{\mathrm{R}h}_{\mathrm{R}g}(\tau+1) = Z_0^{(\mathcal{T})}{}^{\mathrm{R}gh}_{\mathrm{R}g}(\tau), \qquad Z_0^{(\mathcal{T})}{}^{\mathrm{R}h}_{\mathrm{R}g}(-\frac{1}{\tau}) = Z_0^{(\mathcal{T})}{}^{\mathrm{R}g^{-1}}_{\mathrm{R}h}(\tau), \tag{F.41}$$

if the phases caused by the gravitational anomaly are trivial. More schematically,

$$_S \circlearrowleft Z_0^{(\mathcal{T})}{}^{\mathrm{NS}}_{\mathrm{NS}} \overset{T}{\leftrightarrow} Z_0^{(\mathcal{T})}{}^{\mathrm{R}}_{\mathrm{NS}} \overset{S}{\leftrightarrow} Z_0^{(\mathcal{T})}{}^{\mathrm{NS}}_{\mathrm{R}} \circlearrowleft_T \quad , \quad Z_0^{(\mathcal{T})}{}^{\mathrm{R}}_{\mathrm{R}} \circlearrowleft_{T,S} \quad . \tag{F.42}$$

If G contains $(-1)^F$, then we should have

$$Z_0^{(\mathcal{T})}{}^{\mathrm{Y}(-1)^F h}_{\mathrm{X}g}(\tau) = Z_0^{(\mathcal{T})}{}^{\overline{\mathrm{Y}}h}_{\mathrm{X}g}(\tau), \tag{F.43}$$

from the definitions (F.34)–(F.37) of the twisted partition functions, and

$$Z_0^{(\mathcal{T})}{}^{\mathrm{Y}h}_{\mathrm{X}(-1)^F g}(\tau) = Z_0^{(\mathcal{T})}{}^{\mathrm{Y}h}_{\overline{\mathrm{X}}g}(\tau), \tag{F.44}$$

from the discussion above. Here, we introduced the notation

$$\overline{\mathrm{NS}} := \mathrm{R}, \quad \overline{\mathrm{R}} := \mathrm{NS}. \tag{F.45}$$

$\mathrm{GL}(2,\mathbb{Z})$ **transformations of twisted partition functions**

By adding the action of $P : \tau \mapsto -\bar{\tau}$ to the T and S transformations, the $\mathrm{SL}(2,\mathbb{Z})$-action enlarges to the $\mathrm{GL}(2,\mathbb{Z})$-action. Under the P transformation, the twisted partition function transforms as

$$Z_0^{(\mathcal{T})}{}^h_g(-\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}_g} U_h e^{-2\pi\sqrt{-1}\bar{\tau}((L_0-\frac{c}{24})-(\tilde{L}_0-\frac{\tilde{c}}{24}))} \tag{F.46}$$

$$= \left[\mathrm{Tr}_{\mathcal{H}_g} e^{2\pi\sqrt{-1}\tau((L_0^\dagger-\frac{c}{24})-(\tilde{L}_0^\dagger-\frac{\tilde{c}}{24}))} U_h^\dagger\right]^* \tag{F.47}$$

$$= \left[\mathrm{Tr}_{\mathcal{H}_g} U_{h^{-1}} e^{2\pi\sqrt{-1}\tau((L_0-\frac{c}{24})-(\tilde{L}_0-\frac{\tilde{c}}{24}))}\right]^* \tag{F.48}$$

$$= \left[Z_0^{(\mathcal{T})}{}^{h^{-1}}_g(\tau)\right]^*, \tag{F.49}$$

where in the second equation, we simply used $\mathrm{Tr}M = [\mathrm{Tr}M^\dagger]^*$, and in the third equation, we used the fact that L_0 is Hermitian and U_h is unitary. It then follows that the orbifold partition function $Z^{(\mathcal{T}/G)}(\tau) = \sum_{g,h\in G} Z_0^{(\mathcal{T})}{}^h_g(\tau)$ satisfies $Z^{(\mathcal{T}/G)}(-\bar{\tau}) = Z^{(\mathcal{T}/G)}(\tau)^*$. Similarly, for a fermionic twisted partition functions $Z_0^{(\mathcal{T})}{}^{\mathrm{Y}h}_{\mathrm{X}g}(\tau)$ ($\mathrm{X},\mathrm{Y} \in \{\mathrm{NS},\mathrm{R}\}$), we have $Z_0^{(\mathcal{T})}{}^{\mathrm{Y}h}_{\mathrm{X}g}(-\bar{\tau}) = [Z_0^{(\mathcal{T})}{}^{\mathrm{Y}h^{-1}}_{\mathrm{X}g}(\tau)]^*$.

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