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On anomalies and fermionic unitary operators

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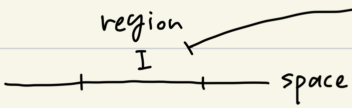
1. Anomalies of fin. grp. sym. of 1+1 d QFT

obstruction to gauging, orbifold

1.1. Bosonic case

Hamiltonian formalism [phys.: Else-Nayak (2014), Seifnashri (2023) (lattice) ; math: Müger (2004) (AQFT, local cfn. net)]

· 1+1d QFT

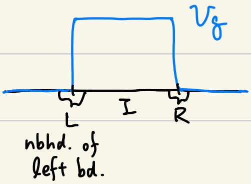


$\mathcal{A}(I) = (\text{op. alg. on } I)$

$= \{\text{op.} \curvearrowright \text{Hilb. sp. } \mathcal{H}|_I\}$

$\mathcal{A} := \bigcup_{I \subset \mathbb{R}} \mathcal{A}(I)$

· sym. action of $g \in G$ on I (G: fin. grp. sym.)



autom. $\rho_g : \mathcal{A} \to \mathcal{A}$

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ \cup & & \cup \\ O & \longmapsto & V_g O (V_g)^{-1} \end{array}$$

· fusion op.

$$\rho_g \rho_h (O) = U_{g,h} \rho_{gh}(O) (U_{g,h})^{-1}$$

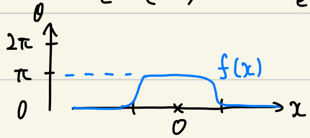
$$U_{g,h} = \underset{\substack{\uparrow \\ \mathcal{A}(L)}}{U^L_{g,h}} \cdot \underset{\substack{\uparrow \\ \mathcal{A}(R)}}{U^R_{g,h}} \quad \text{unitary "fusion op."}$$

$$V_g V_h = U_{g,h} V_{gh}.$$

If $O \in \mathcal{A}(I')$ s.t. $I' \cap L = \emptyset$, $I' \cap R = \emptyset$, then by locality, $\rho_g \rho_h (O) = \rho_{gh}(O)$.

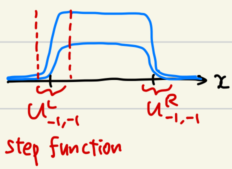
e.g. $G = \mathbb{Z}_2 = \{e^{i0}, e^{i\pi}\} < U(1)$

$$\rho_{e^{i\pi}}(O) = V_{e^{i\pi}}\, O\, (V_{e^{i\pi}})^{-1}$$



$$V_{e^{i\pi}} \sim e^{i\int f(x) J(x)\, dx}$$

$$\rho_{e^{i\pi}}\rho_{e^{i\pi}}(O) = U^L_{-1,-1} U^R_{-1,-1}\, O\, (U^R_{-1,-1})^{-1} (U^L_{-1,-1})^{-1}$$



· anomaly

For any $O \in \mathcal{A}(L)$, $O \in \mathcal{A}(I')$, $\begin{matrix} I' \cap L = \emptyset \\ I' \cap R = \emptyset \end{matrix}$ だと単に $U_{h,k}$ と $\rho_{hk}(O)$ は commute.

$$\rho_g(\rho_h \rho_k(O)) = \rho_g(U_{h,k}\, \rho_{hk}(O)\, U_{h,k}^{-1})$$

$$= \rho_g(U_{h,k})\, U_{g,hk}\, \rho_{ghk}(O)\, U_{g,hk}^{-1}\, \rho_g(U_{h,k})^{-1}$$

$$(\rho_g \rho_h)(\rho_k(O)) = U_{g,h}\, \rho_{gh}(\rho_k(O))\, U_{g,h}^{-1}$$

$$= U_{g,h}\, U_{gh,k}\, \rho_{ghk}(O)\, U_{gh,k}^{-1}\, U_{g,h}^{-1}$$

$\Downarrow$ $\mathcal{A}(L)$: factor $[U_{g,h} U_{gh,k}]^{-1} \rho_g(U_{h,k}) U_{g,hk}$ commutes with $^\forall O \in \mathcal{A}(L)$

$^\exists \alpha : G \times G \times G \to U(1)$ s.t.

$$\rho_g(U_{h,k})\, U_{g,hk} = \alpha(g,h,k)\, U_{g,h}\, U_{gh,k} . \quad \cdots (\star)$$

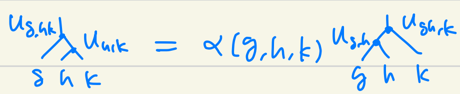
· ρ_ℓ on BHS of $(\star)$ $\rightsquigarrow$ $\delta\alpha = 0$. (abusing notation! $\alpha = e^{2\pi i \alpha}$)

(pentagon identity) $\alpha(g,h,k)\, \alpha(\ell g,h,k)^{-1} \alpha(\ell,gh,k)\, \alpha(\ell,g,hk)^{-1} \alpha(\ell,g,h) = 1$

· redefinition $\tilde{U}_{g,h} := \tau(g,h) U_{g,h}$ $\rightsquigarrow$ $\tilde{\alpha} = \alpha + \delta\tau$

In this way, a cohomology class $[\alpha] \in H^3(G; U(1))$ "anomaly" is associated to the QFT.

obstruction to gauging の説明



If $[\alpha] = 0$, then G is anomaly-free.

1.2. Fermionic case

1+1d $\left\{\begin{matrix}\text{bosonic}\\ \text{fermionic}\end{matrix}\right\}$ anomalies are classified by $\left\{\begin{matrix}H^3(G;U(1))\\ SH(G) \text{ "supercohomology"}\end{matrix}\right.$

$$SH(G) \overset{\text{as a set}}{=} H^3(G;U(1)) \times H^2(G;\mathbb{Z}_2) \times H^1(G;\mathbb{Z}_2)$$

(e.g. $G=\mathbb{Z}_2$ $\quad \mathbb{Z}_8 \quad\quad \mathbb{Z}_2 \quad\quad \mathbb{Z}_2 \quad\quad \mathbb{Z}_2$)

bosonic part

fermionic parity of fusion op.

nontriv. when # real ferm. is odd (not considered in our study)

When $O^L \in \mathcal{A}(L)$,

$|O| \in \{\pm 1\}$: ferm. parity of O

$$\begin{aligned}\rho_g \rho_h (O^L) &= U^L_{g,h} U^R_{g,h}\, \rho_{gh}(O^L) (U^R_{g,h})^{-1} (U^L_{g,h})^{-1} \\ &= |U^R_{g,h}|\,|\rho_{gh}(O^L)|\; U^L_{g,h}\, \rho_{gh}(O^L) (U^L_{g,h})^{-1}\end{aligned}$$

$\rho_g(\bullet) = V_g \bullet V_g^{-1}$ $\quad V_g$: bosonic $\left(\begin{matrix}\text{shrink } I \to 0\\ V_g \to id\end{matrix}\right)$

$\Rightarrow |U^R_{g,h}| = |U^L_{g,h}|,\; |\rho_{gh}(O^L)| = |O^L|$

$$= |U^L_{g,h}|\,|O^L|\; U^L_{g,h}\, \rho_{gh}(O^L)$$

$\nu(g,h) := |U^L_{g,h}|$

- Discussion similar to (☆) $\rightsquigarrow$ $\delta\nu = 0$.
- redefinition $\tilde{V}_g = \Sigma^L_g \Sigma^R_g V_g$ $\rightsquigarrow$ $\tilde{\nu} = \nu + \delta\xi$ $\quad (\xi(g) := |\Sigma^L_g|)$

$\therefore\; [\nu] \in H^2(G;\mathbb{Z}_2)$

($\delta\alpha = 0$ and $\alpha \sim \alpha + \delta\tau$ are modified by ν and ξ.)

$(\delta\alpha)^2 = \nu \cdot \nu$

2. Values of α and ν for $G \subset U(1)$

It is often the case that

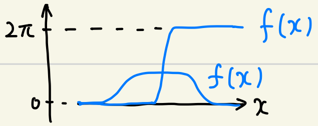
the cohomology class $[(\alpha,\nu)] \in SH(G)$ is known

but the values of cocycles α, ν are not known.

$\rightsquigarrow$ cause problems in calculation of the partition functions of orbifolds (gauged theory)

In the case of $G \subset U(1)$, we succeeded in calculating α, ν.

To do so, we introduce $u(f) := e^{i\int f(x) J(x) dx}$



$J(x)$: level-k $U(1)$ current

$$[J(x), J(y)] = k \frac{i}{2\pi} \partial_y \delta(x-y)$$

In particular, to calculate α, ν, we need the commutator γ

$$u(f)\, u(g) = \underbrace{\gamma(f,g)}_{\in\, U(1)}\, u(g)\, u(f).$$

because we need $\rho_g(U_{n,k}) = V_g U_{n,k} V_g^{-1}$ for $(\circledast)$.

Main difficulty

compactified (because $f(-\infty) = f(\infty)$)

When $f : S^1 \to U(1)$ has non-zero winding number,

it is difficult to make $u(f)$ well-defined.

$f : \mathbb{R} \to U(1)$ だと $\int f\, dx = \infty$ だし
$f : S^1 \to U(1)$ とすると S^1 を $[0, 2\pi]$ などで prm. したときに必ず不連続な jump が発生

Strategy 1 : Introduce patches to S^1 and make $u(f)$ well-def.

Strategy 2 : Impose some requirements from physics on γ, and determine γ uniquely.

requirements

$$\mathcal{F} := \{ S^1 \to U(1) \text{ conti.} \}$$

$\gamma : \mathcal{F} \times \mathcal{F} \to U(1)$ satisfies

- bilinear and alternating $\gamma(f,f) = 0$.
- locality $\gamma(f,g) \in \{\pm 1\}$ if $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$.
- $\gamma(f_0, g_0) = \exp\left[2\pi i \frac{k}{2} \int_{S^1} (f_0 g_0' - g_0 f_0')\, dx\right]$

  for f_0, g_0 : winding number 0.

Both strategies lead to the same results.

winding number

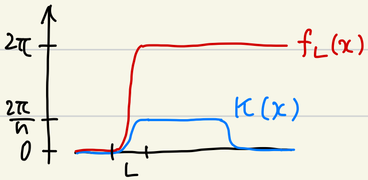
$$\boxed{\gamma(f,g) = \exp 2\pi i \frac{k}{2} \left[\int_0^{2\pi} (f g' - g f')\, dx + f(0)\, w_g - w_f\, g(0) \right]}$$

const. func. (nontriv. elem. of $\mathbb{Z}_2 \subset U(1)$ on entire space)

<u>e.g.</u>
$$u(\tfrac{1}{2})\, u(f)\, u(\tfrac{1}{2})^{-1} = \gamma(\tfrac{1}{2}, f)\, u(f)$$
$$= (-1)^{k w_f}\, u(f)$$

$\rightsquigarrow$ $(-1)^F = u(\frac{1}{2})$, $u(f)$ is fermionic if k and w_f odd.

$\mathbb{Z}_n \subset U(1)$



$$P_a(O) = u(a\kappa)\, O\, u(a\kappa)^{-1}$$
$$0 \le a < n$$

- fusion op.

$$U^L_{b,c} = U(f_L)^{\gamma(b,c)}$$

$$\gamma(b,c) := \begin{cases} 0 & (b+c<n) \\ 1 & (b+c \ge n) \end{cases}$$ carry 繰り上がり

$\Downarrow$ (☆)

$$\alpha(a,b,c) = \exp 2\pi i \left[\frac{k}{2}\frac{a}{n}\gamma(b,c)\right]$$

$k=2$
$\rightsquigarrow \alpha(a,b,c)$ is a rep. of the generator of $H^3(\mathbb{Z}_n; U(1)) \simeq \mathbb{Z}_n$.

$$(-1)^F U^L_{b,c} (-1)^F = \nu(b,c)\, U^L_{b,c}$$

$\Downarrow$

$$\nu(b,c) = (-1)^{k\gamma(b,c)}$$

future directions :

- other abelian case ? $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \subset U(1) \times U(1)$ etc.
- non-ab. case ? $Co_0 \subset Spin(24)$ etc.