On Two-Dimensional Chiral Conformal Field Theories with Sporadic Finite Simple Group Symmetries

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Outline of my dissertation

Part I. Review of Basic Concepts

Part II. Review of Moonshine Phenomena

Part III. Symmetry of Lattice CFT
based on [Okada 2412.19430, JHEP**06**(2025)208]

Part IV. The Conway Orbifold of Duncan's Supermoonshine Module
based on [Albert, Kaidi, Lin, Okada, Tachikawa in preparation]

Part V. Conclusion
Part VI. Appendices

This talk focuses on Part IV.

Plan of this talk

- Introduction
  - What we want to do (Slides 3–5)
  - Why we want to do it (Slides 6–9)
- Review of orbifolds (Slides 10–13)
  - construction, partition function, anomaly
- Compute the Conway orbifold partition function
  - Step 1. Conjugacy classes of commuting pairs (Slide 14)
  - Step 2. Twisted partition functions of Duncan's module (Slides 15–18)
  - Step 3. Anomaly-canceling phases (Slides 19–22)
- Future directions (Slide 23)

What we want to do

(In this talk, we always consider CFTs on 2d torus.)
We want to compute the partition function (the Witten index)

$$Z^{((V^{f\natural})^{\otimes 24}/\mathrm{Co}_1)}(\tau)$$

of a certain orbifold theory.

In general, the orbifold $\mathcal{T}/G$ is a G-invariant 2d CFT constructed from a 2d CFT $\mathcal{T}$ with non-anomalous finite group symmetry G, and its partition function is

$$Z^{(\mathcal{T}/G)}(\tau) = \sum_{[(g,h)]\in\{gh=hg\}/\sim} \frac{|[(g,h)]|}{|G|}\beta(g,h) Z^{(\mathcal{T})}{}^{h}_{g}(\tau).$$

Step 1. List the conjugacy classes of commuting pairs $\{(g,h) \in G \times G \mid gh = hg\} \;/\; (g,h) \sim (kgk^{-1}, khk^{-1})$.

Step 2. Compute the twisted partition functions $Z^{(\mathcal{T})}{}^{h}_{g}(\tau)$.

Step 3. Determine the phases $\beta(g,h)$ to trivialize the anomalous phases. (Note: more fundamental problem than the discrete torsion.)

What we want to do

Step 1. List the conjugacy classes $[(g,h)]$ of commuting pairs.

Step 2. Compute the twisted partition functions $Z^{(\mathcal{T})}{}^h_g(\tau)$.

Step 3. Determine the phases $\beta(g,h)$ to trivialize the anomalous phases.

Each step contains difficulties.

- The difficulties of Step 1 and Step 2 are specific to our theory $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$.
- The difficulty of Step 3 is more essential and also exists in general orbifolds.

What we want to do

Step 1. List the conjugacy classes $[(g,h)]$ of commuting pairs.

Step 2. Compute the twisted partition functions $Z^{(\mathcal{T})}{}^h_g(\tau)$.

Step 3. Determine the phases $\beta(g,h)$ to trivialize the anomalous phases.

(Example) eight real chiral fermions $\psi^{\otimes 8}$ / $\mathbb{Z}_2$ symmetry
The anomaly of one real chiral fermion is the generator of $SH(\mathbb{Z}_2) \cong \mathbb{Z}_8$.

$$Z^{(\psi^{\otimes 8}/\mathbb{Z}_2)}(\tau) = \frac{1}{2}\sum_{g,h\in\mathbb{Z}_2} \beta(g,h) Z^{(\psi^{\otimes 8})}{}^h_g(\tau)$$
$$= \frac{1}{2\eta(\tau)^4}\left(\theta_3(\tau)^4 - \theta_4(\tau)^4 - \theta_2(\tau)^4 \pm \theta_1(\tau)^4\right) \quad {\scriptstyle (\theta_1(\tau)=0)}$$

In this case, we can determine all the phases only from
the modular invariance $Z(\tau+1) \propto Z(\tau)$, $Z(-\frac{1}{\tau}) \propto Z(\tau)$ on $Z(\tau)$.

In more general orbifolds, determining the phases is more difficult.
(In fact, there seems no well-established general method.)

Motivation (Why $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$?)

Conjecture (the Stolz–Teichner conjecture) [Stolz, Teichner 1108.0189]

$$\{\text{2d } \mathcal{N}=(0,1) \text{ SQFTs of degree } n\}/\sim \;\cong\; \mathrm{TMF}_n \quad (n \in \mathbb{Z})$$

"topological modular forms"

- $$\begin{array}{ccc} \mathrm{SQFT}_n \xrightarrow{\sim} \mathrm{TMF}_n & \to & \left\{\text{modular forms of weight } \tfrac{n}{2} \;\; \begin{smallmatrix}\mathbb{Z}\text{-coefficient} \\ \text{weakly holomorphic}\end{smallmatrix}\right\} \\ \Downarrow \!\!\! \cup & & \cup \\ \mathcal{T} & \longmapsto & \text{elliptic genus } \eta(\tau)^n Z^{(\mathcal{T})}(\tau) \end{array}$$

  $$Z^{(\mathcal{T})}(\tau) = \mathrm{Tr}_{\mathcal{H}^{(\mathcal{T})}_{\mathrm{R}\tilde{\mathrm{R}}}}\left[(-1)^{F+\tilde{F}} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\tilde{L}_0 - \frac{c_R}{24}}\right]_{q=e^{2\pi i \tau}}$$

- n specifies the gravitational anomaly of SQFT.
  For an SCFT of central charge (c_L, c_R), $n = 2(c_R - c_L)$.

Assuming this conjecture, we can extract nontrivial statements on the space $\mathrm{SQFT}_\bullet$ of SQFTs, by translating the properties of $\mathrm{TMF}_\bullet$. Conversely, verifying such statements on the $\mathrm{SQFT}_\bullet$ side serves as a test for the Stolz–Teichner conjecture.

Motivation (Why $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$?)

For example, consider translating the following property of $\mathrm{TMF}_\bullet$:

Fact (576-periodicity of $\mathrm{TMF}_\bullet$)

There exists a *periodicity element* $X \in \mathrm{TMF}_{-24^2=-576}$ such that

$$X\cdot : \mathrm{TMF}_n \xrightarrow{\sim} \mathrm{TMF}_{n-576}.$$

If we assume the Stolz–Teichner conjecture, then the existence of the periodicity element X translates to the existence of

an $\mathcal{N}=(0,1)$ SQFT $\mathcal{T}$ of degree $n=-576$
with the elliptic genus (the Witten index) $Z^{(\mathcal{T})}(\tau)=1$.

In particular, if we have

an $\mathcal{N}=1$ $c=288$ chiral SCFT $\mathcal{T}$
with the elliptic genus (the Witten index) $Z^{(\mathcal{T})}(\tau)=1$,

then we only have to put it to the left-moving part ($n=2(c_R-c_L)$).
(The supersymmetry $\mathcal{N}=1$ is imposed so that $Z^{(\mathcal{T})}(\tau)$ is a constant.)

Motivation (Why $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$?)

One candidate of

$$\mathcal{N}=1 \; c=288 \text{ chiral SCFT } \mathcal{T} \text{ with the Witten index } Z^{(\mathcal{T})}(\tau)=1$$

is proposed in [Albert, Kaidi, Lin 2210.14923]. It is constructed from...

The Conway moonshine module (Duncan's module) $V^{f\natural}$

- an $\mathcal{N}=1$ chiral fermionic SCFT of $c=12$. [Duncan math/0502267] [Duncan, Mack-Crane 1409.3829]
  (a $\mathbb{Z}_2$-orbifold of 24 free real chiral fermions)
- has a Conway group Co_1 symmetry.
  Such a big symmetry ($|\mathrm{Co}_1| \sim 4.2\times 10^{18}$) is desirable as follows.
  The Witten index counts the number of Ramond vacuum states.
  Theory with $c=288$ is relatively "big".
  (e.g. The Witten index of $(V^{f\natural})^{\otimes 24}$ is $24^{24}\sim 1.3\times 10^{33}$.)
  Orbifold by a symmetry can reduce the number of states.
- The anomaly of the Co_1 symmetry corresponds to
  the generator of $SH^3(\mathrm{Co}_1)\cong \mathbb{Z}_{24}$. [Johnson-Freyd 1707.08388]

$\to \mathcal{T}=(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$?

Motivation (Why $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$?)

So, we want to compute the Witten index (the partition function)

$$Z^{((V^{f\natural})^{\otimes 24}/\mathrm{Co}_1)}(\tau).$$

In this talk, we provide partial results (Step 1,2: done, Step 3: ongoing), and a conjectural value of this Witten index (> 1).

If $Z^{((V^{f\natural})^{\otimes 24}/\mathrm{Co}_1)}(\tau)$ is still greater than 1,
then we consider to take further orbifold as

$$(V^{f\natural})^{\otimes 24}/(\mathrm{Co}_1 \times A_{24}).$$

($\mathrm{Co}_1 \times S_{24}$ is considered to be anomalous.) [Albert, Kaidi, Lin 2210.14923]

Review of orbifolds (Construction)

$\mathcal{T}$: a 2d CFT periodic in spatial and temporal directions
with non-anomalous finite group symmetry G

Construction of the orbifold theory $\mathcal{T}/G$:

1. Add all the twisted Hilbert spaces $\mathcal{H}_g$ of the states with g-twisted boundary condition in spatial direction ($g \in G$)

$$\mathcal{H}_{\text{tot}} := \bigoplus_{g \in G} \mathcal{H}_g.$$

2. Project $\mathcal{H}_{\text{tot}}$ onto the G-invariant subspace

$$\mathcal{H}^{(\mathcal{T}/G)} := \left(\frac{1}{|G|} \sum_{h \in G} U_h \right) \mathcal{H}_{\text{tot}},$$

where $U_h : \mathcal{H}_{\text{tot}} \to \mathcal{H}_{\text{tot}}$ is the unitary action of $h \in G$.

This $\mathcal{H}^{(\mathcal{T}/G)}$ is the Hilbert space of the orbifold $\mathcal{T}/G$.

Review of orbifolds (Partition function)

Since the Hilbert space is

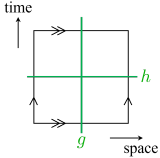
$$\mathcal{H}^{(\mathcal{T}/G)} = \left(\frac{1}{|G|} \sum_{h \in G} U_h \right) \bigoplus_{g \in G} \mathcal{H}_g,$$

the partition function of the orbifold $\mathcal{T}/G$ is

$$Z^{(\mathcal{T}/G)}(\tau) = \frac{1}{|G|} \sum_{g,h \in G} Z_0^{(\mathcal{T})}{}^h_g(\tau),$$

where $Z_0^{(\mathcal{T})}{}^h_g(\tau)$ is the twisted partition function

$$Z_0^{(\mathcal{T})}{}^h_g(\tau) := \mathrm{Tr}_{\mathcal{H}_g}[U_h q^{L_0 - \frac{c_L}{24}} \bar{q}^{\tilde{L}_0 - \frac{c_R}{24}}] \quad (gh = hg),$$



with the effect of anomalous phases already trivialized (next slide).

Review of orbifolds (Anomaly)

Anomaly is an obstacle to the orbifold construction.
For simplicity, in a 2d bosonic theory, the anomaly of a G-symmetry is described by the anomaly 3-cocycle $\alpha : G \times G \times G \to \mathrm{U}(1)$.

- $U_h U_{h'} =$ (phase from α) $U_{hh'}$ on twisted Hilbert spaces $\mathcal{H}_g$.
  $\to P := \frac{1}{|G|} \sum_{h \in G} U_h$ is not a projection to G-inv space ($U_g P \neq P$).
- $Z^{(\mathcal{T})}{}^h_g(\tau + 1) =$ (phase from α) $Z^{(\mathcal{T})}{}^{gh}_g(\tau)$,
  $Z^{(\mathcal{T})}{}^h_g(-\frac{1}{\tau}) =$ (phase from α) $Z^{(\mathcal{T})}{}^{g^{-1}}_h(\tau)$.
  $\to$ The orbifold partition function $Z^{(\mathcal{T}/G)}(\tau)$ is not modular invariant.

If the cohomology class $[\alpha] \in H^3(G; \mathrm{U}(1))$ is trivial,
then using a 2-cochain γ such that $\alpha = d\gamma$,
we can redefine U_g and $Z^{(\mathcal{T})}{}^h_g(\tau)$ so that all the (phases from α) vanish.
So the G-symmetry is said to be non-anomalous.
Notation:

$$
\begin{aligned}
Z_0^{(\mathcal{T})}{}^h_g(\tau) &:= \beta(g,h) Z^{(\mathcal{T})}{}^h_g(\tau) \\
&:= \text{(phase by } \gamma\text{)}\, Z^{(\mathcal{T})}{}^h_g(\tau).
\end{aligned}
$$

Review of orbifolds (Partition function)

Properties of the twisted partition functions $Z_0^{(\mathcal{T})}{}^{h}_{g}(\tau)$:

- $Z_0^{(\mathcal{T})}{}^{h}_{g}(\tau) = 0$ for non-commuting g, h (because $U_h : \mathcal{H}_g \to \mathcal{H}_{hgh^{-1}}$).
- $Z_0^{(\mathcal{T})}{}^{h}_{g}(\tau) = Z_0^{(\mathcal{T})}{}^{khk^{-1}}_{kgk^{-1}}(\tau)$.

$\to$ We only have to compute the twisted partition functions for each conjugacy class of commuting pairs

$$\{(g,h) \in G \times G \mid gh = hg\} \; / \; (g,h) \sim (kgk^{-1}, khk^{-1}).$$

In summary, in order to compute the orbifold partition function

$$Z^{(\mathcal{T}/G)}(\tau) = \sum_{[(g,h)] \in \{gh=hg\}/\sim} \frac{|[(g,h)]|}{|G|} \beta(g,h) Z^{(\mathcal{T})}{}^{h}_{g}(\tau),$$

Step 1. List the conjugacy classes of commuting pairs.

Step 2. Compute the twisted partition functions $Z^{(\mathcal{T})}{}^{h}_{g}(\tau)$.

Step 3. Determine the phases $\beta(g,h)$ to trivialize the anomalous phases.

Let us carry out these steps for $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$.

Step 1. List the conjugacy classes of commuting pairs

We would like to list all the conjugacy classes of commuting pairs

$$\{(g,h) \in \mathrm{Co}_1 \times \mathrm{Co}_1 \mid gh = hg\} \;/\; (g,h) \sim (kgk^{-1}, khk^{-1}).$$

Co_1 is too big and complicated to deal with by hand.
An open-source system GAP is good at handling permutation groups.
So, we represent Co_1 as a permutation group, and pass it to GAP.

(Details:
$\mathrm{Co}_0 = \mathbb{Z}_2.\mathrm{Co}_1$ is the automorphism group of the Leech lattice $\Lambda_{24} \subset \mathbb{R}^{24}$.
→ An element of Co_0 can be represented as a permutation of the 196560 shortest vectors of the Leech lattice Λ_{24}.
→ An element of Co_1 is then a permutation of the $\frac{196560}{2} = 98280$ vectors of $\Lambda_{24}/\{\pm 1\}$.
From the generators of Co_1 represented as permutations,
GAP can list a representative of each conjugacy class of commuting pairs.
We then convert the permutations into matrices $\in \mathrm{SO}(24)$.)

Step 2. Compute the twisted partition functions

We first have to understand the structure of Duncan's module.
Duncan's module $V^{f\natural}$ is constructed as a $\mathbb{Z}_2$-orbifold of
24 free real chiral fermions $\psi^{\otimes 24}$. The details are as follows.

The theory of 24 free real chiral fermions has $\mathrm{Spin}(24)$ symmetry.
The center of $\mathrm{Spin}(24)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle -\hat{1}_{\mathrm{SO}(24)} \rangle \times \langle -1_{\mathrm{Spin}(24)} \rangle$.

$$\mathrm{Spin}(24)$$

$$\swarrow /\langle -\hat{1}_{\mathrm{SO}(24)} \rangle \qquad \searrow /\langle -1_{\mathrm{Spin}(24)} \rangle$$

$$\mathrm{SemiSpin}(24) \qquad\qquad \mathrm{SO}(24)$$

The sectors of this fermionic theory are

	$-1_{\mathrm{Spin}(24)}$ even	$-1_{\mathrm{Spin}(24)}$ odd	
$-\hat{1}_{\mathrm{SO}(24)}$ even	A_0^0	A_1^0	$\rightarrow \quad (V^{f\natural})_{\mathrm{NS}} \curvearrowleft \mathrm{SemiSpin}(24)$
$-\hat{1}_{\mathrm{SO}(24)}$ odd	A_0^1	A_1^1	$\rightarrow \quad (V^{f\natural})_{\mathrm{R}}$
	$\downarrow$	$\downarrow$	
	$(\psi^{\otimes 24})_{\mathrm{NS}}$ $\circlearrowleft$ $\mathrm{SO}(24)$	$(\psi^{\otimes 24})_{\mathrm{R}}$	

Step 2. Compute the twisted partition functions

It is known that $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24}) \subset \mathrm{SO}(24)$ lifts to $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$, and then it projects onto $\mathrm{Co}_1 = \mathrm{Co}_0/\mathbb{Z}_2 \subset \mathrm{SemiSpin}(24)$.

$$\mathrm{Co}_0 \subset \mathrm{Spin}(24)$$

$$\swarrow \qquad\qquad \wr\!\parallel$$

$$\mathrm{SemiSpin}(24) \supset \mathrm{Co}_1 \qquad\qquad \mathrm{Co}_0 \subset \mathrm{SO}(24)$$

In the conformal-weight-$\frac{3}{2}$ subspace of the NS sector $(V^{f\natural})_{\mathrm{NS}}$, there exists one-dimensional invariant subspace under the action of $\mathrm{Co}_1 \subset \mathrm{SemiSpin}(24)$, and it is the $\mathcal{N} = 1$ supercurrent.

	$-1_{\mathrm{Spin}(24)}$ even	$-1_{\mathrm{Spin}(24)}$ odd	
$-\hat{1}_{\mathrm{SO}(24)}$ even		Co_1-inv supercurrent	$\rightarrow \quad (V^{f\natural})_{\mathrm{NS}}$
$-\hat{1}_{\mathrm{SO}(24)}$ odd			$\rightarrow \quad (V^{f\natural})_{\mathrm{R}}$
	$\downarrow$ $(\psi^{\otimes 24})_{\mathrm{NS}}$	$\downarrow$ $(\psi^{\otimes 24})_{\mathrm{R}}$	

Step 2. Compute the twisted partition functions

Now, we would like to calculate the twisted partition functions of $V^{f\natural}$. We already know

- the matrix form $M_g \in \mathrm{SO}(24)$ of $g \in \mathrm{Co}_1$ (Step 1).
- the formula of twisted partition function of $\psi^{\otimes 24}$ twisted by $\mathrm{SO}(24)$.

As the last ingredient, we have to detect
which of the two lifts $\hat{M}_g$ or $-\hat{M}_g \in \mathrm{Spin}(24)$ of $M_g \in \mathrm{SO}(24)$
is in the preimage of $\mathrm{Co}_1 \subset \mathrm{SemiSpin}(24)$.

$$\begin{array}{ccc} & \mathrm{Spin}(24) & \\ & \pm\hat{M}_g \in \mathrm{Co}_0, \mp\hat{M}_g \notin \mathrm{Co}_0 & \\ \swarrow & & \searrow \\ \mathrm{SemiSpin}(24) & & \mathrm{SO}(24) \\ g \in \mathrm{Co}_1, -g \notin \mathrm{Co}_1 & & M_g \in \mathrm{Co}_0 \end{array}$$

(We block-diagonalize $M_g \in \mathrm{SO}(24)$ into 12 two-dimensional rotations by angles $\theta_1, \ldots, \theta_{12}$. We have to detect θ_i or $\theta_i + 2\pi$ for one i.)

Step 2. Compute the twisted partition functions

To detect which lift is correct, we used the fact that
the correct lift $\pm\hat{M}_g \in \mathrm{Co}_0$ preserves the supercurrent,
whereas the wrong one reverses the sign of the supercurrent.

(Details:

1. Represent $\hat{M}_g \in \mathrm{Spin}(24)$ as a 2^{11}-dim matrix $\hat{M}_g^{2^{11}}$ on the conformal-weight-$\frac{3}{2}$ subspace of $(V^{f\natural})_{\mathrm{NS}}$ (the positive chiral spinor representation of $\mathrm{Spin}(24)$).
2. Determine the supercurrent G in the conformal-weight-$\frac{3}{2}$ subspace as a basis of the 1-dim intersection of the invariant subspaces under the generators $\hat{A}^{2^{11}}$ and $\hat{B}^{2^{11}}$ of Co_0.
3. See if $\pm\hat{M}_g^{2^{11}} G = \pm G$ or $\pm\hat{M}_g^{2^{11}} G = \mp G$.)

We succeeded in computing the twisted partition functions $Z^{(V^{f\natural})}{}^h_g(\tau)$ of Duncan's module $V^{f\natural}$, up to the anomalous phases ($\to$ Step 3).

Step 3. Determine the phases to trivialize the anomaly

The cohomology class of the anomaly of the Co_1 symmetry of $V^{f\natural}$ is known to be the generator of $SH(\mathrm{Co}_1) \cong \mathbb{Z}_{24}$. [Johnson-Freyd 1707.08388]

"supercohomology"

Therefore, we can construct the orbifold $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$.
However, we know neither the actual values of the anomaly 3-cocycle $\alpha : \mathrm{Co}_1 \times \mathrm{Co}_1 \times \mathrm{Co}_1 \to \mathrm{U}(1)$, nor the phases β of

$$
\begin{aligned}
Z_0^{((V^{f\natural})^{\otimes 24})}{}^h_g(\tau) &= \beta(g,h)\left(Z^{(V^{f\natural})}{}^h_g(\tau)\right)^{24} \\
&=: \tilde{\beta}(g,h)\left|Z^{(V^{f\natural})}{}^h_g\right|^{24}
\end{aligned}
$$

to trivialize the phases from α.
(There seems no well-established way of calculating them.)

So we attempt to determine the phases $\tilde{\beta}(g,h)$ from some consistency conditions, that is, properties which the twisted partition functions $Z_0^{(\mathcal{T})}{}^h_g(\tau)$ must satisfy.

Step 3. Determine the phases to trivialize the anomaly

There are 7578 conjugacy classes $[(g,h)]$ of commuting pairs, and for each of them, we need to determine $\tilde{\beta}(g,h)$.

- modular transformation ($\mathrm{SL}(2,\mathbb{Z})$ transformation):

$$Z_0^{(\mathcal{T})}{}^h_g(\tau+1) = Z_0^{(\mathcal{T})}{}^{gh}_g(\tau), \quad Z_0^{(\mathcal{T})}{}^h_g(-\frac{1}{\tau}) = Z_0^{(\mathcal{T})}{}^{g^{-1}}_h(\tau).$$

  $\rightarrow$ The number of degrees of freedom of $\tilde{\beta}(g,h)$ reduces to 445.

- $\mathrm{GL}(2,\mathbb{Z})$ transformation:

$$Z_0^{(\mathcal{T})}{}^h_g(-\bar{\tau}) = \left[Z_0^{(\mathcal{T})}{}^{h^{-1}}_g(\tau)\right]^*.$$

  $\rightarrow$ Among 445 $\tilde{\beta}(g,h)$, 427 are $\tilde{\beta}(g,h) \in \{+1,-1\}$, and the remaining $\frac{18}{2} = 9$ pairs are complex conjugates of each other.

Step 3. Determine the phases to trivialize the anomaly

- Each twisted Hilbert space $\mathcal{H}_g$ is a representation of the centralizer $C_g := \{h \in G \mid gh = hg\}$.
  So, the twisted partition function $Z_0^{(\mathcal{T})}{}_g^h(\tau)$ must be a character of $h \in C_g$.
  From the orthogonality of the irreducible characters,
  the multiplicity of the a-th irreducible representation χ_a of C_g is

  $$N(g,a) := \frac{1}{|C_g|} \sum_{h \in C_g} \chi_a(h)^* Z_0^{(\mathcal{T})}{}_g^h(\tau).$$

  All the multiplicities $N(g,a)$ must be integers.

  These integrality conditions can determine some of the phases $\tilde{\beta}(g,h)$, but not all.
  As a nontrivial observation, a simple choice $\tilde{\beta}(g,h) = +1$ for any $g, h \in \mathrm{Co}_1$ satisfies all the integrality conditions!

Step 3. Determine the phases to trivialize the anomaly

Conjecture

The choice $\tilde{\beta}(g,h) = +1$ for any $g, h \in \mathrm{Co}_1$ gives the correct value of the orbifold partition function $Z^{((V^{f\natural})^{\otimes 24}/\mathrm{Co}_1)}(\tau)$.

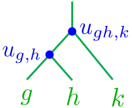
If this is the case, then the orbifold partition function (the Witten index) is $Z^{((V^{f\natural})^{\otimes 24}/\mathrm{Co}_1)}(\tau) = 665834752697050$.

Future directions

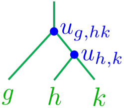
- Determine the phases $\tilde{\beta}(g, h)$ and the Witten index $Z^{((V^{f\natural})^{\otimes 24}/\mathrm{Co}_1)}(\tau)$.
- Compute $Z^{((V^{f\natural})^{\otimes 24}/(\mathrm{Co}_1 \times A_{24}))}(\tau)$. Is it 1?
  Again, we have to determine the phases to trivialize the anomaly.
  These phases are common to $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1$ and $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1 \times A_{24}$, so the integrality conditions on $(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1 \times A_{24}$ might give us additional information on the phases.
- More fundamental approach to determining such phases?
  (cf. In the case of a $\mathrm{U}(1)$ symmetry or its subgroup $\mathbb{Z}_n \subset \mathrm{U}(1)$,
  we can calculate values of the anomaly 3-cocycle $\alpha(g, h, k)$.)
  [Okada, Shimamura, Tachikawa, Yi 2509.02989]

(Backup) Details of anomaly 3-cocycle

The anomaly 3-cocycle $\alpha : G \times G \times G \to \mathrm{U}(1)$ appears in the associativity of the lines implementing the G-action:



$$= \quad \alpha(g,h,k)$$

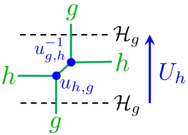


,

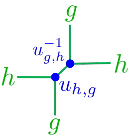
where $u_{g,h} : \mathcal{H}_{g,h} \to \mathcal{H}_{gh}$ is the fusion operator.

We define the action U_h of $h \in G$ on the twisted Hilbert space $\mathcal{H}_g$ and the twisted partition function $Z_g^h(\tau)$ as

$$U_h \; :=$$



$$, \quad Z_g^h(\tau) \; := \; Z\Bigg($$

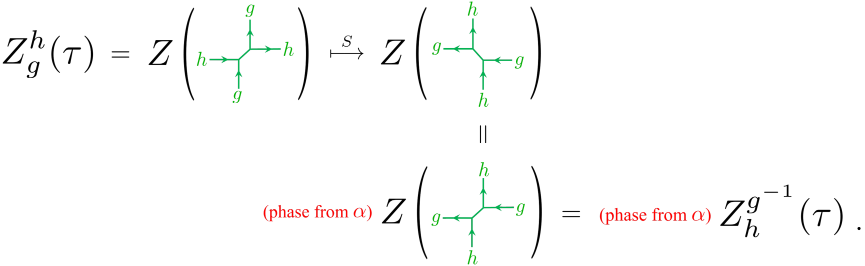


$$\Bigg).$$

(Backup) Details of anomaly 3-cocycle

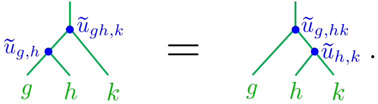
We can see that the anomaly 3-cocycle α is an obstacle to the orbifold; for example, phases from α appear in $U_h U_{h'} =$ (phase from α)$U_{hh'}$, and the modular transformations of $Z_g^h(\tau)$.

(Example) modular S transformation:

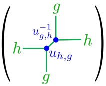


(Backup) Details of anomaly 3-cocycle

If the cohomology class $[\alpha] \in H^3(G; \mathrm{U}(1))$ is trivial,
then using the 2-cochain γ such that $\alpha = d\gamma$,
we define a new fusion operator $\tilde{u}_{g,h} := \gamma(g,h) u_{g,h}$.
Then the phases $\alpha(g,h,k)$ do not appear in the associativity of the new fusion operators $\tilde{u}_{g,h}$



So $\tilde{U}_g$ and $\tilde{Z}_g^h(\tau)$ defined with the new fusion operators $\tilde{u}_{g,h}$ also do not suffer from the phases from α.

Recalling $Z_g^h(\tau) := Z$ (), the new twisted partition function is

$$\tilde{Z}_g^h(\tau) = \frac{\gamma(h,g)}{\gamma(g,h)} Z_g^h(\tau).$$

(${Z_0^{(\mathcal{T})}}_g^h(\tau) = \beta(g,h) {Z^{(\mathcal{T})}}_g^h(\tau)$ in our slides' notation.)

(Backup) Step 2. Detailed description of lifts

The center of $\mathrm{Spin}(24)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle -\hat{1}_{\mathrm{SO}(24)} \rangle \times \langle -1_{\mathrm{Spin}(24)} \rangle$.

$$\begin{array}{ccc} & \mathrm{Spin}(24) & \\ \swarrow /\langle -\hat{1}_{\mathrm{SO}(24)} \rangle & & \searrow /\langle -1_{\mathrm{Spin}(24)} \rangle \\ \mathrm{SemiSpin}(24) & & \mathrm{SO}(24) \end{array}$$

The matrix form of $g \in \mathrm{Co}_1 = \mathrm{Co}_0/\mathbb{Z}_2$ is $\pm M_g \in \mathrm{SO}(24)$.
When we fix one lift $\hat{M}_g \in \mathrm{Spin}(24)$ of $M_g \in \mathrm{SO}(24)$,

$$\mathrm{Spin}(24)$$

$$\begin{array}{r|r} \hat{M}_g \in \mathrm{Co}_0 & (-1_{\mathrm{Spin}(24)} \cdot \hat{M}_g) \notin \mathrm{Co}_0 \\ \hline -\hat{1}_{\mathrm{SO}(24)} \cdot \hat{M}_g \in \mathrm{Co}_0 & -\hat{1}_{\mathrm{SO}(24)} \cdot (-1_{\mathrm{Spin}(24)} \cdot \hat{M}_g) \notin \mathrm{Co}_0 \end{array}$$

or

$$\begin{array}{r|r} (-1_{\mathrm{Spin}(24)} \cdot \hat{M}_g) \in \mathrm{Co}_0 & \hat{M}_g \notin \mathrm{Co}_0 \\ \hline -\hat{1}_{\mathrm{SO}(24)} \cdot (-1_{\mathrm{Spin}(24)} \cdot \hat{M}_g) \in \mathrm{Co}_0 & -\hat{1}_{\mathrm{SO}(24)} \cdot \hat{M}_g \notin \mathrm{Co}_0 \end{array}$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$\mathrm{SemiSpin}(24)$ $\qquad\qquad$ $\mathrm{SO}(24)$

$$g \in \mathrm{Co}_1 \;\Big|\; -g \notin \mathrm{Co}_1 \qquad\qquad \frac{M_g \in \mathrm{Co}_0}{-M_g \in \mathrm{Co}_0}$$

(Backup) Step 2. Twisted partition function of fermions

If M_g and $M_h \in \mathrm{SO}(24)$ commute, then we can simultaneously block-diagonalize them within $\mathrm{SO}(24)$ in the form of

$$\begin{pmatrix} \cos 2\pi\lambda_1 & -\sin 2\pi\lambda_1 & & & \\ \sin 2\pi\lambda_1 & \cos 2\pi\lambda_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi\lambda_{12} & -\sin 2\pi\lambda_{12} \\ & & & \sin 2\pi\lambda_{12} & \cos 2\pi\lambda_{12} \end{pmatrix}.$$

The twisted partition function $Z^{(\psi^{\otimes 24})}{}^{M_h}_{M_g}(\tau)$ of 24 real chiral fermions can be written in terms of the theta function with characteristic $\theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau)$

$$Z^{(\psi^{\otimes 24})}{}^{M_h}_{M_g}(\tau) = \prod_{i=1}^{12} \frac{1}{\eta(\tau)} \theta \begin{bmatrix} \lambda_i^{(g)} \\ \lambda_i^{(h)} \end{bmatrix} (\tau).$$

The difference of the two lifts $\pm \hat{M}_g \in \mathrm{Spin}(24)$ can be implemented as the difference of λ_{12} and $\lambda_{12} + 1$.