On orbifolds of 2d chiral fermionic CFTs by finite non-abelian groups

Masaki Okada, Yuji Tachikawa

Kavli IPMU, the University of Tokyo

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[work in progress]

Message

We want to compute the partition function of the orbifold of the 24-fold tensor product of the Conway moonshine module (Duncan's module) $V^{f\natural}$ by the Conway group Co_1 (anomalous-free)

$$Z^{(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1}(\tau).$$

However, the calculation of an orbifold partition function was much more difficult than we expected.

$$Z^{\mathcal{T}/G}(\tau) = \sum_{[(g,h)]\in\{gh=hg\}/\sim} \frac{\#[(g,h)]}{\#G}\beta(g,h) Z^{\mathcal{T}}_{g,h}(\tau).$$

Step 1. List the conjugacy classes of commuting pairs $\{(g,h)\in G\times G \mid gh=hg\} \;/\; (kgk^{-1},khk^{-1})\sim(g,h)$.

Step 2. Compute the twisted partition functions $Z^{\mathcal{T}}_{g,h}(\tau)$.

Step 3. Determine the phases $\beta(g,h)\in \mathrm{U}(1)$ to cancel the phases coming from anomaly 3-cocycle (which is trivial in cohomology since G must be anomaly-free).

Motivation

Conjecture (the Stolz–Teichner conjecture)

$$\{\text{2d } \mathcal{N} = (0,1) \text{ SQFTs of degree } n \ (= 2(c_R - c_L))\}/\!\sim \ \cong \ \mathrm{TMF}_n$$

[Stolz, Teichner 1108.0189]

The space of topological modular forms TMF_n has 576-periodicity

$$\mathrm{TMF}_{n+576} \ \cong \ \mathrm{TMF}_n.$$

If we believe the Stolz–Teichner conjecture, then the existence of

an $\mathcal{N} = 1$ $c = 288$ chiral SCFT $\mathcal{T}$
with the partition function (the Witten index) $Z^{\mathcal{T}}(\tau) = 1$

is expected.

Motivation

One candidate of

$$\mathcal{N}=1\ c=288 \text{ chiral SCFT } \mathcal{T} \text{ with the Witten index } Z^{\mathcal{T}}(\tau)=1$$

is constructed with...

The Conway moonshine module (Duncan's module) $V^{f\natural}$

- an $\mathcal{N}=1$ chiral fermionic SCFT. [Duncan math/0502267] [Duncan, Mack-Crane 1409.3829]
- the central charge is 12.
  $\to \mathcal{T}=(V^{f\natural})^{\otimes 24}$?
  $\to Z^{(V^{f\natural})^{\otimes 24}}(\tau)=24^{24}$. Too big.
- $V^{f\natural}$ has Co_1 symmetry, and its anomaly corresponds to the generator of $SH^3(B\mathrm{Co}_1)\cong\mathbb{Z}_{24}$. [Johnson-Freyd 1707.08388]
  $\to \mathcal{T}=(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1\ (\times S_{24}$ or $A_{24})$? [Albert, Kaidi, Lin 2210.14923]
  $\to Z^{(V^{f\natural})^{\otimes 24}/\mathrm{Co}_1}(\tau)=$?

More on the Conway moonshine module

The Conway moonshine module $V^{f\natural}$ is constructed as a $\mathbb{Z}_2$-orbifold of 24 real free fermions $\psi^{\otimes 24}$. The details are as follows.

The theory of 24 real free fermions has $\mathrm{Spin}(24)$ symmetry.
The center of $\mathrm{Spin}(24)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle -\hat{1}_{\mathrm{SO}(24)} \rangle \times \langle -1_{\mathrm{Spin}(24)} \rangle$.

$$\mathrm{Spin}(24) \xrightarrow{/\langle -\hat{1}_{\mathrm{SO}(24)} \rangle} \mathrm{SemiSpin}(24) \supset \langle -\bar{1}_{\mathrm{Spin}(24)} \rangle$$

$$\mathrm{Spin}(24) \xrightarrow{/\langle -1_{\mathrm{Spin}(24)} \rangle} \mathrm{SO}(24) \supset \langle -1_{\mathrm{SO}(24)} \rangle$$

The sectors of this fermionic theory are

	$-1_{\mathrm{Spin}(24)}$ even	$-1_{\mathrm{Spin}(24)}$ odd	
$-\hat{1}_{\mathrm{SO}(24)}$ even	A_0^0	A_1^0	$\to (V^{f\natural})_{\mathrm{NS}} \curvearrowleft \mathrm{SemiSpin}(24)$
$-\hat{1}_{\mathrm{SO}(24)}$ odd	A_0^1	A_1^1	$\to (V^{f\natural})_{\mathrm{R}}$
	$\downarrow$ $(\psi^{\otimes 24})_{\mathrm{NS}}$ $\circlearrowleft \mathrm{SO}(24)$	$\downarrow$ $(\psi^{\otimes 24})_{\mathrm{R}}$	

$\langle -\hat{1}_{\mathrm{SO}(24)} \rangle$-orbifold: $(\psi^{\otimes 24}) \rightleftarrows (V^{f\natural})$: $\langle -1_{\mathrm{Spin}(24)} \rangle$-orbifold

More on the Conway moonshine module

The Conway group Co_0 is the automorphism group of a 24-dimensional even self-dual lattice called the Leech lattice.
So $\mathrm{Co}_0 \subset \mathrm{SO}(24)$. It is known that it lifts to $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$, and then it projects to $\mathrm{Co}_1 = \mathrm{Co}_0/\mathbb{Z}_2 \subset \mathrm{SemiSpin}(24)$.

$$
\begin{array}{ccc}
 & \mathrm{Co}_0 \subset \mathrm{Spin}(24) & \\
 \swarrow & & \cong \\
\mathrm{SemiSpin}(24) \supset \mathrm{Co}_1 & & \mathrm{Co}_0 \subset \mathrm{SO}(24)
\end{array}
$$

In the conformal-weight-$\frac{3}{2}$ subspace of the NS sector $(V^{f\natural})_{\mathrm{NS}}$, there exists one-dimensional subspace invariant under the action of $\mathrm{Co}_1 \subset \mathrm{SemiSpin}(24)$, and it is an $\mathcal{N}=1$ supercurrent.

	$-1_{\mathrm{Spin}(24)}$ even	$-1_{\mathrm{Spin}(24)}$ odd	
$-\hat{1}_{\mathrm{SO}(24)}$ even		Co_1-inv supercurrent	$\rightarrow$ $(V^{f\natural})_{\mathrm{NS}}$
$-\hat{1}_{\mathrm{SO}(24)}$ odd			$\rightarrow$ $(V^{f\natural})_{\mathrm{R}}$
	$\downarrow$ $(\psi^{\otimes 24})_{\mathrm{NS}}$	$\downarrow$ $(\psi^{\otimes 24})_{\mathrm{R}}$	

Step 1. List the conjugacy classes of commuting pairs

We have to list all the conjugacy classes of commuting pairs

$$\{(g,h) \in \mathrm{Co}_1 \times \mathrm{Co}_1 \mid gh = hg\} \; / \; (kgk^{-1}, khk^{-1}) \sim (g,h).$$

$\mathrm{Co}_0 = 2.\mathrm{Co}_1$ is the automorphism group of the Leech lattice.
The shortest vectors of the Leech lattice have squared-norm 4, and there are 196560 shortest vectors.
Therefore, an element of Co_0 can be represented as a permutation of 196560 points.
An element of Co_1 is then a permutation of $\frac{196560}{2} = 98280$ points.

An open source system GAP is good at handling permutation groups.
If we input the generators of Co_1 represented as permutations, GAP can list a representative of each conjugacy class of commuting pairs.

Step 2. Compute the twisted partition functions

The twisted partition functions of $V^{f\natural}$ can be calculated from
the twisted partition functions of $\psi^{\otimes 24}$

$$Z^{\psi^{\otimes 24}}_{\mathrm{X}g,\mathrm{Y}h}(\tau), \quad \mathrm{X},\mathrm{Y} = \mathrm{NS} \text{ or } \mathrm{R}, \quad g,h \in \mathrm{Co}_0,$$

where $\mathrm{X}g$ and $\mathrm{Y}h$ denote the spatial and temporal twists respectively.

To obtain the correct twisted partition functions,
we have to consider everything in $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$,
but in practice, $\mathrm{Co}_0 \subset \mathrm{SO}(24)$ as a matrix group is easier to handle,
so we work with $\mathrm{Co}_0 \subset \mathrm{SO}(24)$ as far as possible.

Step 2. Compute the twisted partition functions

As input data, we have the matrix form of $g, h \in \mathrm{Co}_0$ in $\mathrm{SO}(24)$. Basically, we only have to simultaneously block-diagonalize them within $\mathrm{SO}(24)$ as

$$\begin{pmatrix} \cos 2\pi\theta_1 & -\sin 2\pi\theta_1 & & & \\ \sin 2\pi\theta_1 & \cos 2\pi\theta_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi\theta_{12} & -\sin 2\pi\theta_{12} \\ & & & \sin 2\pi\theta_{12} & \cos 2\pi\theta_{12} \end{pmatrix},$$

and plug these angles $2\pi\theta$ into the formula of Jacobi theta function

$$Z^{\psi^{\otimes 2}}_{\mathrm{X}e^{2\pi i\theta_{(g)}},\mathrm{Y}e^{2\pi i\theta_{(h)}}}(\tau) = q^{\theta_{(g)}^2/2}\frac{\theta_{1,2,3,4}(\tau, \theta_{(g)}\tau + \theta_{(h)})}{\eta(\tau)}.$$

Step 2. Compute the twisted partition functions

However, $g \in \mathrm{Co}_0 \subset \mathrm{SO}(24)$ lifts to two elements $\hat{g}, -\hat{g}$ in $\mathrm{Spin}(24)$.
In terms of angles, one is $2\pi\theta$ and the other is $2\pi(\theta+1)$.
One of them is in $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$,
whereas the other is out of $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$.
So we have to detect which of $2\pi\theta$ or $2\pi(\theta+1)$ is the correct one.

This is extremely hard!

There are 445 $\mathrm{SL}(2,\mathbb{Z})$-orbits of twisted partition functions $Z^{V^{f\natural}}_{\mathrm{R}g,\mathrm{R}h}$,
and we could calculate the correct values for 435 out of them
by using some consistency conditions and properties of
$\mathrm{Spin}(24)$-conjugacy classes.
We cannot use them for the rest 10 because of some subtle properties
of their $\mathrm{Spin}(24)$-conjugacy classes.
$\rightarrow$ Use the fact that $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$ preserves the supercurrent?

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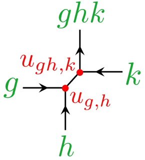
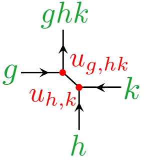
→ Use the fact that $\mathrm{Co}_0 \subset \mathrm{Spin}(24)$ preserves the supercurrent?
→ Succeeded on the train from Hiroshima to Saijo!

Step 3. Determine the phases to trivialize the anomaly

The anomaly 3-cocycle $\alpha : G \times G \times G \to \mathrm{U}(1)$ appears as

$Z\Big($ 

$\Big) \;=\; \alpha(g, h, k) \cdot Z\Big($ 

$\Big)$.

If the cohomology class $[\alpha] \in H^3(G; \mathrm{U}(1))$ is trivial, then we can cancel α by redefining the fusion operators $u_{g,h}$ by $\mathrm{U}(1)$ phases.

(If $[\alpha] \neq 0$, then α cannot be canceled, and this is the obstruction of the modular invariance of the orbifold partition function.
So we cannot take the orbifold.)

Step 3. Determine the phases to trivialize the anomaly

Therefore, when we take an orbifold by non-anomalous group G, we have to sum up the twisted partition functions while multiplying them by the phases to cancel the effect of anomaly cocycle α.

Theoretically, we know that the anomaly of Co_1 symmetry of $(V^{f\natural})^{\otimes 24}$ is trivial at the level of cohomology, but we do not know its values as a 3-cocycle.

It seems that no general techniques to determine the phases have been developed so far.
We might be able to do so by exploiting some consistency conditions such as the modular invariance, the integrality of the coefficients of the partition functions, and so on.

We are still on the way...