On the Symmetry of Lattice CFTs

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based on [arXiv:2412.19430]

Question (for mathematicians)

L : a lattice of rank n.
$\hat{L}$: the central extension

$$1 \to \mathbb{Z}_2 \to \hat{L} \to L \to 0 \quad (1)$$

specified by the 2-cocycle $\varepsilon : L \times L \to \mathbb{Z}_2$ required by physics.
The *lattice VOA* V_L is constructed from this $\hat{L}$.

Question: Is $\mathrm{Aut}(L)$ a subgroup of $\mathrm{Aut}(V_L)$?

It is known that $\mathrm{Aut}(V_L)$ contains $\mathrm{Aut}(\hat{L})$ as a subgroup, and $\mathrm{Aut}(\hat{L})$ is an extension of $\mathrm{Aut}(L)$:

$$1 \to \mathrm{Hom}(L, \mathbb{Z}_2) \to \mathrm{Aut}(\hat{L}) \to \mathrm{Aut}(L) \to 1. \quad (2)$$

So, the problem is whether this (2) splits or not.

* $\mathrm{Aut}(L)$ and $\mathrm{Aut}(\hat{L})$ denote the automorphisms compatible with the bilinear form of L.

Question (for physicists)

From a lattice L, we can construct a chiral CFT V_L whose momentum lattice is L.

The vertex operator $V_k(z)$ $(k \in L)$ is roughly $:e^{ik\cdot X(z)}:$, but we need a correction to modify their commutation relations into the bosonic / fermionic ones. As a result, the cocycle factor $\varepsilon(k,k') \in \mathbb{Z}_2$ appears:

$$V_k(z) \cdot V_{k'}(z') \sim \varepsilon(k,k')(z-z')^{k\cdot k'} V_{k+k'}(z'). \quad (3)$$

It is known that a symmetry $g \in \mathrm{Aut}(L)$ of the lattice can be lifted to a symmetry $\hat{g} \in \mathrm{Aut}(V_L)$ of the CFT, but because of the cocycle factor, the group structure of $\mathrm{Aut}(L)$ is not guaranteed to be preserved by this lift.

Question: Is $\mathrm{Aut}(L)$ a subgroup of $\mathrm{Aut}(V_L)$?

What is the motivation?

In the study of moonshine phenomena, one of the goals is to construct a CFT (VOA) having a specific finite group symmetry. The symmetry of a lattice is often a promising ingredient (e.g. the monstrous moonshine), and in that case, we need to investigate the relation between $\mathrm{Aut}(L)$ and $\mathrm{Aut}(V_L)$ carefully.

For example, the odd Leech lattice CFT was considered to be a nice candidate of a CFT explaining the moonshine between the $c = 24$ $\mathcal{N} = 2$ extremal elliptic genus and the Mathieu group M_{24} [BDFK15], but the following result reveals that there are more difficulties than expected in the previous work.

Results

- When L is the odd Leech lattice,
  $M_{24}, M_{23} \subset \mathrm{Aut}(L)$ is not a subgroup of $\mathrm{Aut}(V_L)$.
- When L is the Leech lattice,
  $\mathrm{Co}_0 = \mathrm{Aut}(L)$ is not a subgroup of $\mathrm{Aut}(V_L)$. [Gri73]
- Our method only needs the data of the lattice and the action of $\mathrm{Aut}(L)$. No need of heavy group theory!

Method (for mathematicians)

Take generators of $\mathrm{Aut}(L)$, say a and b, and their relations

$$a^2 = 1,\ b^3 = 1,\ (ba)^{23} = 1,\ \ldots. \quad (4)$$

Assume that (2) has a group homomorphism section $S : \mathrm{Aut}(L) \to \mathrm{Aut}(\hat{L})$. Then

$$S(a)^2 = 1,\ S(b)^3 = 1,\ (S(b)S(a))^{23} = 1,\ \ldots. \quad (5)$$

Since S is a homomorphism, the values of $S(g)(\hat{e}_i)$ $(i = 1, \ldots, n)$ completely determine the value of $S(g)(\hat{k})$ for any $\hat{k} \in \hat{L}$, where $\hat{e}_i \in \hat{L}$ is a lift of a basis $e_i \in L$.

Therefore, (5) translates to the equations w.r.t. $S(a)(e_i)$ and $S(b)(e_i)$, but they do not have a solution (by just a linear algebra; use computer).

* Dictionary
$$\left\{ \begin{array}{ccc} S(g) \in \mathrm{Aut}(\hat{L}) & \hat{k} \in \hat{L} & S(g)(\hat{k}) \\ \updownarrow & \updownarrow & \updownarrow \\ \hat{g} \in \mathrm{Aut}(V_L) & V_k(z) \in V_L & \zeta_g(k) V_{g(k)}(z) \end{array} \right\}$$

Method (for physicists)

The symmetry $\hat{g} \in \mathrm{Aut}(V_L)$ must satisfy

$$\hat{g}(V_k(z)) \cdot \hat{g}(V_{k'}(z')) \sim \varepsilon(k,k')(z-z')^{k\cdot k'} \hat{g}(V_{k+k'}(z')). \quad (6)$$

This (6) translates to the condition on the phase factor $\zeta_g(k) \in \mathbb{Z}_2$ of $\hat{g}(V_k(z)) = \zeta_g(k) V_{g(k)}(z)$ as

$$\varepsilon(k,k')\zeta_g(k+k') = \zeta_g(k)\zeta_g(k')\varepsilon(g(k), g(k')). \quad (7)$$

This (7) determines $\zeta_g(k)$ for any $k \in L$ from the values of $\zeta_g(e_i)$ $(i = 1, \ldots, n)$ where e_i is a basis of L.

Assume $\mathrm{Aut}(L)$ is a subgroup of $\mathrm{Aut}(V_L)$. Then any relation $g_1 \cdots g_m = 1$ in $\mathrm{Aut}(L)$ translates to $\hat{g}_1 \circ \cdots \circ \hat{g}_m(V_k(z)) = V_k(z)$ in $\mathrm{Aut}(V_L)$, and further

$$\zeta_{g_1}(g_2 \cdots g_m(k)) \cdots \zeta_{g_{m-1}}(g_m(k))\zeta_{g_m}(k) = 1. \quad (8)$$

However, we found by computer a set of relations in $\mathrm{Aut}(L)$ such that there are no values $\zeta_{g_j}(e_i)$ satisfying (8) for them.

References

[Gri73] R. L. Griess, Jr., *Automorphisms of extra special groups and nonvanishing degree 2 cohomology*, Pacific J. Math. **48** (1973) 403–422.

[BDFK15] N. Benjamin, E. Dyer, A. L. Fitzpatrick, and S. Kachru, *An extremal $\mathcal{N} = 2$ superconformal field theory*, J. Phys. A **48** (2015) 495401, arXiv:1507.00004 [hep-th].