Moonshine Phenomena and Symmetries of lattice CFTs

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Outline

1. Moonshine
   - 1-1. What is monstrous moonshine?
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   - 1-3. More moonshine phenomena
2. My research

(3. Group extension) if time permits

1-1. What is monstrous moonshine?

Here is the modular j-function.

$$j(q) = q^{-1} + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + \cdots$$

This function is important in elliptic curve theory, class field theory, ...

($j(q)$ establishes

$$\left\{ \begin{array}{l} \text{isom. classes} \\ \qquad \text{of tori} \end{array} \right\} \xleftrightarrow{1:1} \mathbb{C}\,. \;)$$

In 1978, John McKay noticed $196884 = 1 + 196883$.

Furthermore, J. G. Thompson observed

$$j(q) = q^{-1} + 744 + \underbrace{196884}_{\substack{\| \\ 1 + 196883}} q + \underbrace{21493760}_{\substack{\| \\ 1 + 196883 \\ + 21296876}} q^2 + \underbrace{864299970}_{\substack{\| \\ 2\times 1 + 2\times 196883 \\ + 21296876 + 842609326}} q^3 + \cdots$$

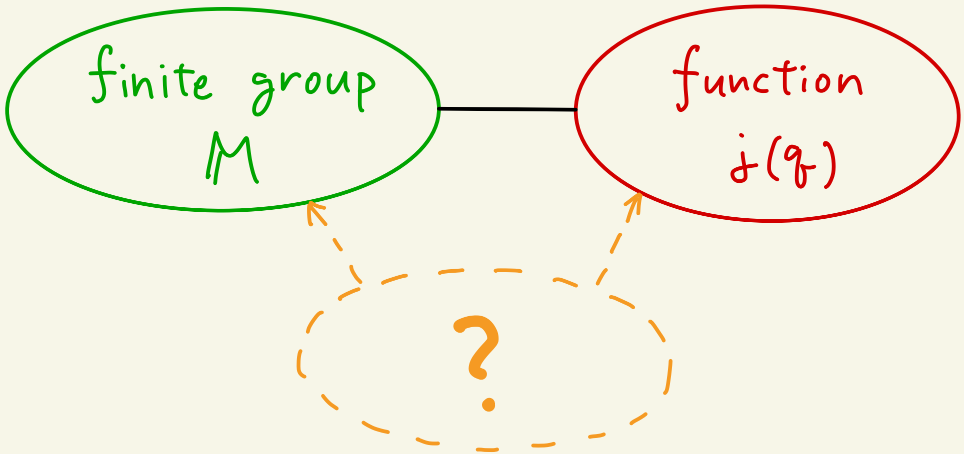
Observation

$$\boxed{\text{Data (irrep. dim.) of the monster group } \mathbb{M} \text{ appear in the coefficients of } j(q)\,!}$$

($\mathbb{M}$ is the largest sporadic group.) "monstrous moonshine"

It implies that

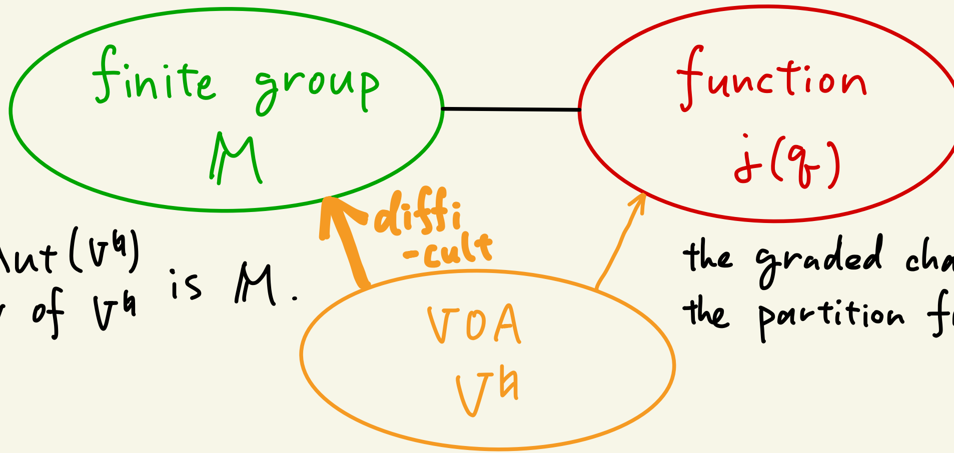
something which explains the observation is hidden.



(there would exist some nice infinite-dim. graded vector space
which is an $\mathbb{M}$-module (rep. of $\mathbb{M}$)
whose graded character is $j(q)$.)

In fact, Frenkel, Lepowski, Meurman (1983-88) constructed it!

It was a VOA (Vertex Operator Algebra) called moonshine module $V^\natural$.



$\mathrm{Aut}(V^\natural)$ / symmetry of $V^\natural$ is $\mathbb{M}$.

the graded character / the partition function of $V^\natural$ is $j(q) - 744$.

(VOA also gives a mathematical formulation of 2d CFT (Conformal Field Theory) in physics.)

What does $V^\natural$ look like? How $\boxed{\mathrm{Aut}(V^\natural) = \mathbb{M}}$ was proved?

1-2. How was it proved?

The VOA $V^\natural$ was constructed from the Leech lattice $\Lambda \subset \mathbb{R}^{24}$ and an additional information called cocycle factors taking values ± 1.

$$\text{lattice } \Lambda \xrightarrow{\text{cocycle factors}} V_\Lambda \xrightarrow{(\mathbb{Z}_2 \text{ orbifold})} V^\natural$$

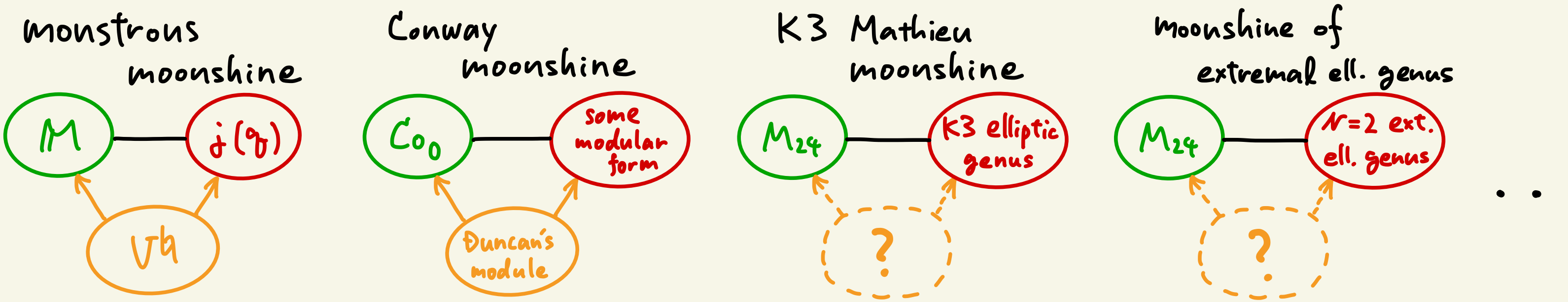
As a result, the symmetry group $\mathrm{Aut}(V^\natural)$ of $V^\natural$ is a certain combination of $\mathrm{Aut}(\Lambda)$ and $(\mathbb{Z}_2)^{24}$ (and $\mathbb{Z}_2$).

$$\mathrm{Aut}(\Lambda) = Co_0 \xrightarrow{\text{combined with } (\mathbb{Z}_2)^{24}} \mathrm{Aut}(V_\Lambda) \xrightarrow{(\text{combined with } \mathbb{Z}_2)} \mathrm{Aut}(V^\natural)$$

A careful analysis of these combinations reveals $\boxed{\mathrm{Aut}(V^\natural) = \mathbb{M}}$.

1-3. More moonshine phenomena

As a result of interaction between mathematics and physics, relations similar to that of $\mathbb{M}$ and $j(q)$ were observed in more examples of finite groups and functions.



A nice underlying object is already known in some cases, but not yet found in many cases.

2. My research

- Ultimate goal is to find nice underlying objects.
- I developed an easy way to analyze this combination for more general lattices.

$$\underset{\text{a lattice}}{\mathrm{Aut}(L)} \xrightarrow{\text{combined with } (\mathbb{Z}_2)^{\mathrm{rank}\,L}} \mathrm{Aut}(V_L)$$

As a result of the analysis for the odd Leech lattice O_{24}, I found a gap overlooked in a previous research on a certain moonshine.

[arXiv:2412.19430]

- work in progress: Is Duncan's module useful for K3 moonshine?

3. Group extension

ways of combining two groups N and G

= ways of introducing a multiplication law in the set $\{(n,g) \mid n \in N,\ g \in G\}$.

direct product   the easiest way!

$$(n,g)\cdot(n',g') = (nn',\ gg')$$

generalize ↓   ↑ special case ($G \curvearrowright N$ is trivial)

semidirect product

$$(n,g)\cdot(n',g') = (n\,g(n'),\ gg')$$

generalize ↓   ↑ special case (the exact seq. splits)

group extension of G by N

$$(n,g)\cdot(n',g') = (n\,g(n')\,\varepsilon(g,g'),\ gg')$$

exact sequence

$$1 \to N \to \circledast \to G \to 1$$

direct prod.

semidirect prod.

group extension

N is always a subgroup of a group extension of G by N.

However, G is <u>not</u> a subgroup of it <u>in general</u>.

(G is a subgroup if and only if it is a semidirect product.)

It was already known that

a specific group extension of $\mathrm{Aut}(L)$ by $(\mathbb{Z}_2)^{\mathrm{rank}\, L}$

is a subgroup of $\mathrm{Aut}(V_L)$.

My analysis showed that $\mathrm{Aut}(O_{24})$ is <u>not</u> a subgroup of $\mathrm{Aut}(V_{O_{24}})$.

(the group extension does not split.)