On the Mathieu group symmetry of an odd Leech lattice CFT

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September 16, 2024

[work in progress]

Summary

a lattice $L \rightsquigarrow$ the lattice VOA V_L (CFT with momentum lattice L)

$\mathrm{Aut}(L)$ is **not** a subgroup of $\mathrm{Aut}(V_L)$ in general

because of **the cocycle factor**. Instead,

A group extension of $\mathrm{Aut}(L)$ is a subgroup of $\mathrm{Aut}(V_L)$.

[Frenkel, Lepowsky, Meurman 1988]
[Dong, Nagatomo math/9808088]

In fact, we proved

- For $L =$ odd Leech lattice,
  $M_{24}, M_{23} \subset \mathrm{Aut}(L)$ are not subgroups of $\mathrm{Aut}(V_L)$.
- For $L =$ Leech lattice,
  $\mathrm{Co}_0 = \mathrm{Aut}(L)$ is not a subgroup of $\mathrm{Aut}(V_L)$. [Griess 1973]

$\rightarrow$ Some moonshine phenomena seem more mysterious than we think.

Background: Moonshine Phenomena

The elliptic genus of a 2d $\mathcal{N}=2$ SCFT (chiral for simplicity)
is defined as $\mathrm{tr}_{\mathcal{H}_{\mathrm{R}}}(-1)^F q^{L_0-\frac{c}{24}} y^{J_0}$.

If we decompose

the elliptic genus $Z_{\mathrm{ext}}(q,y)$ of a certain 2d $\mathcal{N}=2$ SCFT
(called the $\mathcal{N}=2$ index-4 extremal elliptic genus)

into

the characters $\mathrm{ch}_{h,Q}(q,y)$ of

the $c=24$ $\mathcal{N}=2$ superconformal algebra (SCA),

$$Z_{\mathrm{ext}} = \mathrm{ch}_{1,4} + 47\mathrm{ch}_{1,0} + 23\mathrm{ch}_{2,4} + 2024\mathrm{ch}_{2,3} + 14168\mathrm{ch}_{2,2} + \cdots .$$

In the coefficients,
the dimensions of irreps of the Mathieu group M_{24} appear:

1, $47 = 45 + 1 + 1$, 23, 2024, $14168 = 10395 + 3520 + 253$, ...

"Moonshine phenomenon"

$\to$ CFT $= \bigoplus_i$ (rep of $M_{24})_i \otimes$ (irrep of $\mathcal{N}=2$ SCA$)_{h_i,Q_i}$?

Background: Moonshine Phenomena

Question

Does there exist a VOA-like object V containing an $\mathcal{N}=2$ SCA, whose elliptic genus is the extremal elliptic genus $Z_{\mathrm{ext}}(q,y)$, and whose symmetry group $\mathrm{Aut}(V)^{\mathcal{N}=2}$ preserving the $\mathcal{N}=2$ SCA contains the Mathieu group M_{24} as a subgroup?

Examples of Moonshine phenomena:

Mostrous Moonshine	weight-0 mod. func. (J-function $J(q)$)	Virasoro algebra	Monster group $\mathbb{M}$	Monster VOA $V^\natural = V_{\mathrm{Leech}}/\mathbb{Z}_2$
Conway Moonshine	$\frac{\theta_{E_8}(\tau)\eta(\tau)^8}{\eta(\tau/2)^8\eta(2\tau)^8} - 8$	$\mathcal{N}=1$ SCA	Conway group Co_0	Duncan SVOA $V^{s\natural}$
K3 Mathieu Moonshine	weight-0 index-1 weak Jacobi form	$\mathcal{N}=4$ SCA	Mathieu group M_{24}	?
above Question	weight-0 index-4 weak Jacobi form Z_{ext}	$\mathcal{N}=2$ SCA	Mathieu group M_{24}	?

⋮

Background: Idea to explain M_{23} by [BDFK]

Discussion by [Benjamin, Dyer, Fitzpatrick, Kachru 1507.00004]:

1. odd Leech lattice L : the unique rank-24 odd self-dual lattice without vector of square-length 2
   $\rightsquigarrow$ the fermionic CFT whose NS sector V_L consists of states
   $$\alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_n}_{-m_n} |k\rangle \quad (\mu_i = 1, ..., 24,\ m_i \in \mathbb{Z}_{>0},\ k \in L).$$
2. We can find a $c = 24$ $\mathcal{N} = 2$ SCA $\{T(z), J(z), G^{\pm}(z)\}$ in V_L, such that the elliptic genus is $Z_{\mathrm{ext}}(q, y)$.
   Its $U(1)$ current $J(z)$ is in the direction of the first coordinate: $J(z) \sim \partial X^1(z)$.
3. The isometry group $\mathrm{Aut}(L)$ of L contains M_{24} as a permutation group of the 24 coordinates.
   Its subgroup stabilizing one coordinate is M_{23}.
4. Therefore, the automorphism group $\mathrm{Aut}(V_L)$ of V_L contains M_{24}, and its subgroup M_{23} preserves the $\mathcal{N} = 2$ SCA...?

   Here is a problem.

[(not covered in this talk) There also seems to be a problem in the choice of supercurrents $G^{\pm}(z)$ in Step 2, but it is amendable by constructing the $\mathcal{N}$=2 SCA from ternary codes [Gaiotto, Johnson-Freyd 1811.00589].]

Cocycle Factor

Some literature on moonshine argues that

$\mathrm{Aut}(L)$ is a subgroup of $\mathrm{Aut}(V_L)$,

but it is **not** the case in general because of **the cocycle factor**.

The cocycle factor:

The vertex operator $V_k(z)$ $(k \in L)$ of V_L is roughly $: e^{ik\cdot X(z)} :$,
but we need to modify the commutation relation

$$: e^{ik\cdot X(z)} : \cdot : e^{ik'\cdot X(w)} : = (-1)^{k\cdot k'} : e^{ik'\cdot X(w)} : \cdot : e^{ik\cdot X(z)} :$$

to the desired one of the vertex operators

$$V_k(z) \cdot V_{k'}(w) = (-1)^{|k|^2|k'|^2} V_{k'}(w) \cdot V_k(z).$$

As a result, **the cocycle factor** $\varepsilon : L \times L \to \mathbb{Z}_2$ appears:

$$V_k(z) \cdot V_{k'}(w) \sim \varepsilon(k, k')(z - w)^{k\cdot k'} V_{k+k'}(w).$$

(The cocycle condition follows from the associativity of $V_k(z)$.)

(Outline) Proof that $\mathrm{Aut}(L)$ is not a subgroup of $\mathrm{Aut}(V_L)$

An isometry $g \in \mathrm{Aut}(L)$ must lift to an automorphism $\hat{g} \in \mathrm{Aut}(V_L)$ as

$$(*) \quad \hat{g}(V_k(z)) \cdot \hat{g}(V_{k'}(w)) \sim \varepsilon(k,k')(z-w)^{k\cdot k'}\hat{g}(V_{k+k'}(w)).$$

$(*)$ translates to the condition on the phase factor $\zeta_g(k) \in \mathbb{Z}_2$ or $U(1)$ of $\hat{g}(V_k(z)) = \zeta_g(k) V_{g(k)}(z)$ as

$$\varepsilon(k,k')\zeta_g(k+k') = \zeta_g(k)\zeta_g(k')\varepsilon(g(k),g(k')).$$

This condition determines $\zeta_g(k)$ for any $k \in L$

from the values of $\zeta_g(e_1), \ldots, \zeta_g(e_{24})$ where $\{e_i\}_i$ is a basis of L.

If $\mathrm{Aut}(L)$ is a subgroup of $\mathrm{Aut}(V_L)$,

then any relation $g_1 \cdots g_n = 1$ in $\mathrm{Aut}(L)$

translates to $\hat{g}_1 \circ \cdots \circ \hat{g}_n(V_k(z)) = V_k(z)$ in $\mathrm{Aut}(V_L)$,

and further $(**)$ $\zeta_{g_1}(g_2 \cdots g_n(k)) \cdots \zeta_{g_{n-1}}(g_n(k))\zeta_{g_n}(k) = 1$.

However, there exist a set of relations in $\mathrm{Aut}(L)$ such that

there are no values $\zeta_{g_i}(e_1), \ldots, \zeta_{g_i}(e_{24})$ satisfying $(**)$ for them.

Mathematical Formulation

The lattice VOA V_L is contructed from
the central extension $\hat{L}$ of L by $\mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1 \rangle$:

$$1 \ \to \ \mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1 \rangle \ \to \ \hat{L} \ \to \ L \ \to \ 0,$$

associated to the 2-cocycle $\varepsilon : L \times L \to \mathbb{Z}_2$.

(An element (κ^m, k) of $\hat{L}$ corresponds to $(-1)^m V_k(z)$,
and the multiplication in $\hat{L}$ is like $V_k(z) \cdot V_{k'}(z) \sim \varepsilon(k, k') V_{k+k'}(z)$.)

$\mathrm{Aut}(V_L)$ contains $\mathrm{Aut}(\hat{L})$ as a subgroup,
and $\mathrm{Aut}(\hat{L})$ is a group extension of $\mathrm{Aut}(L)$ [Frenkel, Lepowsky, Meurman]:

$$(\sharp) \quad 1 \to \mathrm{Hom}(L, \mathbb{Z}_2) \to \mathrm{Aut}(\hat{L}) \to \mathrm{Aut}(L) \to 1.$$

$\mathrm{Hom}(L, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{24}$ corresponds to the choice of $\zeta_g(e_1), \ldots, \zeta_g(e_{24})$.

Therefore, whether $\mathrm{Aut}(L)$ is a subgroup of $\mathrm{Aut}(\hat{L})$ or not
is equivalent to whether the short exact sequence $(\sharp)$ splits or not.

Results and Future Work

We proved

- For $L =$ odd Leech lattice,
  $M_{24}, M_{23} \subset \mathrm{Aut}(L)$ are not subgroups of $\mathrm{Aut}(V_L)$.
- For $L =$ Leech lattice,
  $\mathrm{Co}_0 = \mathrm{Aut}(L)$ is not a subgroup of $\mathrm{Aut}(V_L)$. [Griess 1973]

Instead, for $L =$ odd Leech lattice,

$\mathrm{Aut}(V_L)$ contains $(\mathbb{Z}_2)^{24}.M_{24}$ as a subgroup.

Its subgroup stabilizing one coordinate is $(\mathbb{Z}_2)^{23}.M_{23}$.
However, the $\mathcal{N} = 2$ SCA by [BDFK] is **not** preserved by $(\mathbb{Z}_2)^{23}.M_{23}$!
(The supercurrents $G^{\pm}(z)$ are subject to nontrivial action of $(\mathbb{Z}_2)^{23}$.)

Future work:
Why the dimensions of irreps of M_{24} appear? Even M_{23} is nontrivial.
(the subalgebra of the $\mathcal{N} = 2$ SCA consisting of only $T(z)$ and $J(z)$? another CFT?)