# Asymptotic density of states in 2d CFTs with non-invertible symmetries

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based on a joint work with
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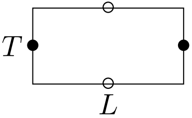
[arXiv:2208.05495[hep-th]]

# 2d diagonal RCFT on torus




$$\mathcal{H}_1 = \bigoplus_a V_a \otimes \overline{V_a}$$

$$Z_1(q) = \mathrm{tr}_{\mathcal{H}_1}\, q^{H/2\pi} = \sum_a \chi_a(q)\overline{\chi_a(q)}$$

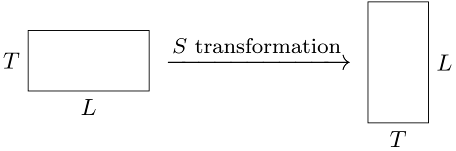
$$(q = e^{-2\pi(T/L)} = e^{-2\pi\beta})$$

the asymptotic density of states in the sector $V_a \otimes \overline{V_a}$:

$$\text{“}\ \frac{\dim V_a \otimes \overline{V_a}}{\dim \mathcal{H}_1}\ \text{”} = \lim_{\beta \to 0} \frac{\mathrm{tr}_{V_a \otimes \overline{V_a}}\, e^{-\beta H}}{\mathrm{tr}_{\mathcal{H}_1} e^{-\beta H}}$$

(ratio of partition function)

# 2d diagonal RCFT on torus




$q' = e^{-2\pi(L/T)} \ll 1$
(low temperature)

$$\chi_a(q) = \sum_b \overline{S_{ba}} \chi_b(q') \quad \sim \quad S_{1a} q'^{-c/24}$$

$\uparrow$ assumption

$$\begin{cases} \chi_1(q') = q'^{-c/24}(1 + \bullet\, q'^{\bullet} + \cdots) \\ \chi_{a\neq 1}(q') = q'^{-c/24}(\bullet\, q'^{\bullet} + \cdots) \end{cases}$$

$$\therefore \quad \mathrm{tr}_{V_a \otimes \overline{V_a}} q^{H/2\pi} = \chi_a(q)\overline{\chi_a(q)} \sim S_{1a}\overline{S_{1a}} q'^{-c/12}$$

$\therefore$ (asymptotic density of $V_a \otimes \overline{V_a}$) $\propto S_{1a}\overline{S_{1a}}$

$\rightarrow$ How to derive such asymptotic densities for more general CFTs with modular tensor category (MTC) symmetry, fusion category (FC) symmetry, ...?

(More fundamentally, what is the counterpart of "the sector $V_a \otimes \overline{V_a}$" for more general CFTs with modular tensor category (MTC) symmetry, fusion category (FC) symmetry, ..?)

# 2d CFT with finite group symmetry $G$

the action of $g \in G$ on $\mathcal{H}_1$

$$U_g : \mathcal{H}_1 \to \mathcal{H}_1$$

$$\mathcal{H}_1 = \bigoplus_{\rho \in \mathrm{Irrep}(G)} \mathcal{H}_1^\rho$$

$$Z_1(q) = \mathrm{tr}_{\mathcal{H}_1}\, q^{H/2\pi}$$

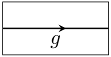
# 2d CFT with finite group symmetry $G$

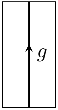
the asymptotic density of states in the sector $\mathcal{H}_1^\rho$ :
(S. Pal and Z. Sun [2004.12557])

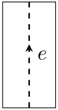
① $P_1^\rho := \frac{\dim\rho}{|G|}\sum_{g\in G}\overline{\chi_\rho(g)}U_g$ is the projection $\mathcal{H}_1 \twoheadrightarrow \mathcal{H}_1^\rho$.

($\because$ the character orthogonality $\sum_{g\in G}\overline{\chi_\rho(g)}\chi_{\rho'}(g) = |G|\delta_{\rho,\rho'}$)

② $Z_1^\rho(q) := \mathrm{tr}_{\mathcal{H}_1^\rho}\, q^{H/2\pi} = \mathrm{tr}_{\mathcal{H}_1} P_1^\rho q^{H/2\pi}$

$$= \frac{\dim\rho}{|G|}\sum_{g\in G}\overline{\chi_\rho(g)}$$


$$= \frac{\dim\rho}{|G|}\sum_{g\in G}\overline{\chi_\rho(g)}$$


$$\sim \frac{\dim\rho}{|G|}\qquad\overline{\chi_\rho(e)}$$


$$= \frac{(\dim\rho)^2}{|G|}Z_1(q')$$

$\therefore$ (asymptotic density of $\mathcal{H}_1^\rho$) $\propto (\dim\rho)^2$

# finite group symmetry vs diagonal RCFT

- finite group $G$ :
  invertible fusion category consisting of

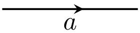
$$
U_g = \xrightarrow[\;g\;]{} \; : \quad
\begin{array}{ccc}
\mathcal{H}_1^\rho & \to & \mathcal{H}_1^\rho \\
\Cup & & \Cup \\
|\psi\rangle & \mapsto & \rho(g)|\psi\rangle
\end{array}
$$

- diagonal RCFT :
  modular tensor category consisting of

$$
\text{the Verlinde lines} \xrightarrow[\;a\;]{} \; : \quad
\begin{array}{ccc}
V_b \otimes \overline{V_b} & \to & V_b \otimes \overline{V_b} \\
\Cup & & \Cup \\
|\psi\rangle & \mapsto & \dfrac{S_{ab}}{S_{1b}}|\psi\rangle
\end{array}
$$

# 2d diagonal RCFT revisited

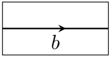
the asymptotic density of states in the sector $V_a \otimes \overline{V_a}$ :

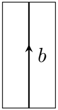
① $P_1^{(a,a)} := \sum_b S_{1a}\overline{S_{ba}}$ 

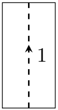
is

the projection $\mathcal{H}_1 \twoheadrightarrow V_a \otimes \overline{V_a}$.

② $\mathrm{tr}_{V_a \otimes \overline{V_a}}\, q^{H/2\pi} = \mathrm{tr}_{\mathcal{H}_1} P_1^{(a,a)} q^{H/2\pi}$

$= \sum_b S_{1a}\overline{S_{ba}}$ 

$= \sum_b S_{1a}\overline{S_{ba}}$ 


$\sim \quad S_{1a}\overline{S_{1a}}$ 

$= S_{1a}\overline{S_{1a}} Z_1(q')$

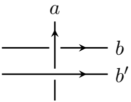
$\therefore$ (asymptotic density of $V_a \otimes \overline{V_a}$) $\propto S_{1a}\overline{S_{1a}}$

For general MTC, just define the sector $\mathcal{H}_1^{(a,a)} := P^{(a,a)}\mathcal{H}_1$.

# 2d diagonal RCFT: twisted sectors

the projections for
the $a$-twisted Hilbert space $\mathcal{H}_a = \bigoplus_{c,d} N_{ad}^c V_c \otimes \overline{V_d}$ :

$$P_a^{(c,d)} := \sum_{b,b'} S_{1c}\overline{S_{cb}} S_{1d}\overline{S_{db'}}$$



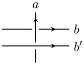
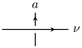
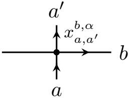

$$: \mathcal{H}_a \twoheadrightarrow N_{ad}^c V_c \otimes \overline{V_d}$$

$\longrightarrow$ (asymptotic density of $N_{ad}^c V_c \otimes \overline{V_d}$) $\propto N_{ad}^c S_{1c} S_{1d}$

For general MTC, just define the sector $\mathcal{H}_a^{(c,d)} := P_a^{(c,d)} \mathcal{H}_a$.

# What is the sector $\mathcal{H}_a^{(c,d)}$ for general MTC?

$\to \mathcal{H}_a^{(c,d)}$ is the $a$-twisted sector
of the sector $\mathcal{H}^{(c,d)}$ which transforms under
the irreducible representation labeled by $(c,d)$
of the "tube algebra" of the MTC.

| symmetry category | MTC $\mathcal{M}$ | FC $\mathcal{C}$ |
|---|---|---|
| Drinfeld center | $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \boxtimes \mathcal{M}^{\mathrm{op}}$ | $\mathcal{Z}(\mathcal{C})$ |
| projector $P_a^\mu : \mathcal{H}_a \twoheadrightarrow \mathcal{H}_a^\mu$ ($\mu \in$ Drinfeld center) | $P_a^{(c,d)} := \sum_{b,b'} S_{1c}\overline{S_{cb}}S_{1d}\overline{S_{db'}}$   $\mathcal{H}_a^{(c,d)} := P_a^{(c,d)}\mathcal{H}_a$ | $P_a^\mu := \sum_\nu S_{1\mu}\overline{S_{\mu\nu}}$   $\mathcal{H}_a^\mu := P_a^\mu \mathcal{H}_a$ |
| tube algebra | $\Big\{$   $\Big\vert\ a, a', b \in \mathcal{C},\ x_{a,a'}^{b,\alpha} \in \mathrm{Hom}(b \otimes a, a' \otimes b) \Big\}$ | |
| $\mathcal{H}^\mu \curvearrowleft$ irrep $\mu$ of tube alg | $\mathcal{H}^{(c,d)} = \bigoplus_a \mathcal{H}_a^{(c,d)}$ | $\mathcal{H}^\mu = \bigoplus_a \mathcal{H}_a^\mu$ |
| asymptotic density of $\mathcal{H}_a^\mu$ | $\propto N_{ad}^c S_{1c} S_{1d}$ | $\propto \langle \mu, a \rangle \dim \mu$ |