Towards Hodge Theoretic Characterizations of 2d Rational SCFTs

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based on joint works with
A. Kidambi and T. Watari [arXiv:2205.10299[hep-th]]
T. Watari [arXiv:2212.13028[hep-th]]

2d Rational CFT

At some special points in a moduli space of CFT, the symmetry of CFT is enhanced.

$$\left(\begin{array}{l} \text{the chiral algebra } \mathcal{A} \text{ is larger than the Virasoro algebra} \\ \rightarrow \# \text{ (primary fields (irrep.) of } \mathcal{A}) < \infty \end{array} \right)$$

$\rightarrow$ Rational CFT

known examples of 2d RCFT

- It is determined when a torus-target CFT is rational.
  e.g. S^1 target with radius R such that $\frac{R^2}{\alpha'} \in \mathbb{Q}$
- Gepner models $\rightarrow$ some Calabi-Yau targets

How many more RCFTs? characterization by target geometry?

Gukov-Vafa Conjecture

Conjecture [Gukov Vafa '02]

For a Ricci-flat Käler target $(M; G, B)$,

the SCFT is rational $\overset{?}{\Longleftrightarrow}$ M and its mirror W are of CM-type with the same CM field

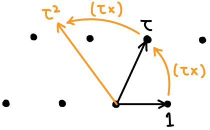
In [Kidambi Watari MO '22, Watari MO '22],

- we pointed out some need for refinement of this statement.
- we examined this statement for $M = T^4$.
  $\rightarrow$ "RCFT $\Rightarrow$ CM" but the converse is not true.
- By adding one more condition,
  we can achieve a necessary and sufficient condition.

Complex Multiplication

CM-type : a generalized notion of
an elliptic curve with complex multiplication.

$T^2_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$



If ${}^\exists a, b \in \mathbb{Z}$ such that $\tau^2 = a\tau + b$,
$(\tau\times)$ is a linear map $\mathbb{Z} \oplus \tau\mathbb{Z} \to \mathbb{Z} \oplus \tau\mathbb{Z}$.
$(\quad T^2_\tau \quad \to \quad T^2_\tau \quad)$

Otherwise,
$(\tau\times)$ is not such a linear map.

$\rightsquigarrow (\otimes\mathbb{Q})$

If ${}^\exists D \in \mathbb{Z}_{<0}$ such that $\tau \in \mathbb{Q}(\sqrt{D})$,

$$\mathrm{End}(H^1(T^2_\tau;\mathbb{Q}))^{\mathrm{Hdg}} \cong \mathbb{Q}(\sqrt{D}).$$

"T^2_τ is an elliptic curve with complex multiplication."
"The Hodge structure on $H^1(T^2_\tau;\mathbb{Q})$ is of CM-type."

(Otherwise, $\mathrm{End}(H^1(T^2_\tau;\mathbb{Q}))^{\mathrm{Hdg}} \cong \mathbb{Q}$.)

Supporting Evidence of the GV Conjecture

[Moore '98, Gukov Vafa '02]

- elliptic curves $M = T^2$

  RCFT $\Leftrightarrow$ $^{\exists}D \in \mathbb{Z}_{<0}$ such that $\tau, \rho \in \mathbb{Q}(\sqrt{D})$ $\quad M \cong T^2_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, $\rho = \int_M \frac{B+i\omega}{(2\pi)^2\alpha'}$

  $M \cong T^2_\tau$ and $W \cong T^2_\rho$ are of CM-type with $\mathbb{Q}(\sqrt{D})$.

- the Fermat quintic $M = \{z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0\} \subset \mathbb{P}^4$

  $\cdots$ target space of the Gepner model $(\mathrm{Min}^{\mathcal{N}=2,\mathrm{diag}}_{c=9/5})^{\otimes 5}/\mathbb{Z}_5$

  $\cdots$

$$\underline{M}\quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \boxed{1 \;\; 101 \;\; 101 \;\; 1}\;\} = \begin{array}{c} \boxed{1\;\;1\;\;1\;\;1} \\ + \\ 0\;\;\boxed{2\;\;2}\;\;0 \end{array} \times 50 \qquad \underline{W}\quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The **Hodge structures** are of CM-type with $\mathbb{Q}(\zeta_5)$.

GV Conjecture Revisited

Some need for refinement of the GV conjecture:

- Choice of complex structure
  - $\mathcal{N} = (1,1)$ SCFT $\leftsquigarrow (M; G, B)$
  - the Hodge structure on $H^*(M;\mathbb{Q}) \leftsquigarrow (M; I)$
- Existence and choice of mirror
  In general, for $\mathcal{N} = (2,2)$ SCFT $(M; G, B; I)$,
  - its mirror $(M_\circ; G_\circ, B_\circ; I_\circ)$ does not always exist.
  - even if exists, not necessarily unique.
- Which Hodge structure should be of CM-type?
  There can be multiple Hodge structures on $H^*(M;\mathbb{Q})$.

T^4

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

CY 4-fold

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{3,3} & & 0 & & \\ & 0 & & h^{3,2} & & h^{2,3} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{1,3} & & 1 \\ & 0 & & h^{2,1} & & h^{1,2} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Results 1 [Kidambi Watari MO '22, Wendland '00(Sep), Meng Chen '05]

- $(T^4; G, B)$ gives RCFT

  $\Rightarrow$ • $^\exists$complex structure I such that

  - a mirror $(T^4_\circ; G_\circ, B_\circ; I_\circ)$ exists and for any mirror,
  - the Hodge structures on $H^*(T^4; \mathbb{Q})$ and
    the Hodge structures on $H^*(T^4_\circ; \mathbb{Q})$ are of CM-type
    and Hodge isomorphic.

Results 2 [Kidambi Watari MO '22, Watari MO '22]

- The converse is not true:
  even if we impose all the conditions on $(T^4; G, B)$,
  there remain some non-RCFT cases.

e.g. $T^4 \simeq T^2_{\tau_1} \times T^2_{\tau_2}$

	$T^2_{\tau_1}$	$T^2_{\tau_2}$
RCFT	$\tau_1, \rho_1 \in \mathbb{Q}(\sqrt{p_1})$	$\tau_2, \rho_2 \in \mathbb{Q}(\sqrt{p_2})$
non-RCFT	$\tau_1 \in \mathbb{Q}(\sqrt{p_1}), \rho_1 \in \mathbb{Q}(\sqrt{p_2})$	$\tau_2 \in \mathbb{Q}(\sqrt{p_2}), \rho_2 \in \mathbb{Q}(\sqrt{p_1})$

- An additional condition
  - There exists an isogeny $\phi : T^4_\circ \to T^4$ such that $\phi^* : H^1(T^4; \mathbb{Z}) \to H^1(T^4_\circ; \mathbb{Z})$ satisfies

$$\phi^*|_{\Gamma_b^\vee} = \mathrm{id}|_{\Gamma_b^\vee}$$

    where $\Gamma_b \subset H_1(T^4; \mathbb{Z})$ denotes the "non-T-dualized directions".

  eliminates the non-RCFT cases, and
  gives a necessary and sufficient condition for RCFT of $M = T^4$.